FINAL EXAM, M340L, F94 — Norton's section

Directions: Do all problems and show your work clearly. Put a box around your final answer for each problem. *You must show your work and justify your answers to receive credit.* No books, notes, or calculators are permitted, except one 8.5 x 11 sheet of notepaper with your own notes. Use the space provided; continue on the back if you need more space.

There are ten questions, each worth 10 points; 100 points total.

1. Use row reduction to find all solutions of the following system. Express your answer in vector parametric form.

$$\begin{aligned} 2x + 2z &= 4 \\ y + 2z + w &= 4 \\ x + z + w &= 4. \end{aligned}$$

2. Let W be the subspace of R^4 spanned by the vectors $(1, 0, 1, 0), (2, 1, -1, 0)$, and $(1, 2, -5, 0)$.

(a) Find a basis for W.

(b) What is the dimension of W?

(c) Let $\mathbf{x} = (6, 2, 0, 0)$. Find $[\mathbf{x}]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis found in part (a).

3. Let

$$A = \begin{bmatrix} 10 & 20 & 30 \\ 6 & 12 & 24 \\ 12 & 15 & 1994 \end{bmatrix}.$$

(a) Find the determinant of A.

(b) Find the determinant of $2A$.

4. Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Find A^{-1}.

(b) Use your answer to (a) to solve $A\mathbf{x} = (1, 2, 5)$ for $\mathbf{x}$.

5. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & -5 & 0 \end{bmatrix}.$$

(a) Find bases for the row space, column space, and null space of A.

(b) Define rank. What is the rank of A?

6. Let A be an $n \times n$ matrix. List at least ten different conditions that are all equivalent to the statement "A is invertible". (Do not include this statement itself as one of the ten.)

(Scoring: 1 point each; 10 points maximum.)

7. Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) For each eigenvalue, find a corresponding eigenvector.

(c) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. (Check your answer.)

8. Consider the system

$$
\begin{aligned}
x + 4y &= 1 \\
x + y &= 2 \\
x + y &= 3.
\end{aligned}
$$

(a) Demonstrate that this system is inconsistent.

(b) Write down the corresponding *normal equations* for this system.

(c) Find the best solution(s) in the sense of least squares.

(d) Compute the least squares error for the solutions found in part (c).

9. Let C_k denote the number of cats in central park at the end of k months, and M_k the corresponding number of mice. The populations of cats and mice have been determined to evolve according to the equations

$$
\begin{aligned}
C_{k+1} &= .2C_k + .9M_k \\
M_{k+1} &= -.6C_k + 2.3M_k.
\end{aligned}
$$

It is known that the eigenvalues of the matrix

$$
\begin{bmatrix} .2 & .9 \\ -.6 & 2.3 \end{bmatrix}
$$

are 2 and 0.5, with corresponding eigenvectors $(1, 2)$ and $(3, 1)$, respectively.

(a) Suppose that initially there are 20 cats and 100 mice. Find a formula, in terms of k, for the number of cats C_k after k months.

(b) Some time later a survey finds that there are 200,000 cats in the park. Approximately how many cats will there be one month later? Justify.

10. True or False. (If true, justify; if false, provide a counterexample and explanation.)

(a) Every 2×2 matrix can be diagonalized.

(b) If A and B have the same eigenvalues, then A and B are similar.

(c) If $A^2 = 0$, then $A = 0$.

(d) If A is a 5×7 matrix and $\mathrm{Nul}(A)$ has dimension three, then the system $A\mathbf{x} = \mathbf{b}$ will be inconsistent for certain vectors $\mathbf{b}$.