1. The matrices
\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & -3 \\ 2 & 6 & 5 & 37 \\ 1 & -1 & 2 & 23 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
are row-equivalent.

a) Find a basis for \( \text{Col}(A) \). What is \( \dim(\text{Col}(A)) \)?

b) Find a basis for \( \text{Nul}(A) \). What is \( \dim(\text{Nul}(A)) \)?

c) Find a basis for \( \text{Row}(A) \). What is \( \dim(\text{Row}(A)) \)?

d) \( M_{2,2} \) is the space of \( 2 \times 2 \) matrices. Let \( V \) be the subspace of \( M_{2,2} \) spanned by \( \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 6 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}, \begin{pmatrix} 12 & -3 \\ 37 & 23 \end{pmatrix} \right\} \). Find a basis for \( V \).

2. On \( \mathbb{R}^3 \), let \( \mathcal{E} \) be the standard basis and let \( \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \).

Let \( \mathbf{v} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} \).

a) Compute the change-of-basis matrices \( P_{\mathcal{E} \mathcal{B}} \) and \( P_{\mathcal{B} \mathcal{E}} \)

b) Compute \( [\mathbf{v}]_{\mathcal{B}} \).

c) In \( P_2 \), let \( \mathcal{C} = \{1 + t + t^2, 2 + 3t + t^2, 1 + t + 2t^2\} \), and let \( \mathbf{w} = 5 - 2t + 3t^2 \). Find \( [\mathbf{w}]_{\mathcal{C}} \). (Justify your answer!)

3. Let \( A = \begin{pmatrix} 6 & 5 \\ -5 & 0 \end{pmatrix} \) and let \( B = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 4 & -1 \\ 3 & -1 & 0 \end{pmatrix} \).

a) Find the characteristic equation of \( A \).

(b) Find the eigenvalues of \( A \) (you do not need to find the eigenvectors).

c) The eigenvalues of \( B \) are 1, 2 and 3. Find the corresponding eigenvectors.
(Note: you may get some simple fractions in your calculations, but if you get any truly ugly denominators, you’ve made a mistake.)

4. a) Find a \( 2 \times 2 \) matrix with eigenvalues 1 and 3, and with corresponding eigenvectors \( \begin{pmatrix} 2 \\ 3 \end{pmatrix} \) and \( \begin{pmatrix} 3 \\ 5 \end{pmatrix} \).

(b) Is \( \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \) diagonalizable? Why or why not?
5. True/false. Just mark each statement with a T (or TRUE) or an F (or FALSE). You do not need to justify your answers, and partial credit will not be given.

a) If a square matrix has determinant zero, then its null space is at least 1-dimensional.
b) The plane $x_1 + 2x_2 + 3x_3 = 6$ is a subspace of $\mathbb{R}^3$.
c) If an $m \times n$ matrix has rank $k$, then its null space has dimension $m - k$.
d) If $A$ is a $4 \times 7$ matrix, then the dimension of $Col(A)$ equals the dimension of $Row(A)$.
e) If $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are bases for a vector space $V$, then $P_{BD} = P_{CD}P_{BC}$.
f) The geometric multiplicity of an eigenvalue is at least as big as the algebraic multiplicity of that eigenvalue.
g) If the characteristic equation of a square matrix $A$ is $(\lambda - 1)^3(\lambda + 2) = 0$, then $\lambda = 1$ is an eigenvalue with algebraic multiplicity 3.
h) If the characteristic equation of a real matrix $A$ has complex roots, then there is no basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$.
i) If $\mathcal{B}$ is a basis consisting of eigenvectors of $A$, then $[A]_\mathcal{B}$ is diagonal.
j) If $A$ is a $5 \times 5$ matrix with eigenvalues 11, 25, 32 and 47, and if the geometric multiplicity of $\lambda = 11$ is 2, then $A$ is diagonalizable.