M346 Final Exam, December 15, 2009

1) The matrix $A = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 2 & -1 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 3 & 3 & 5 & 7 \end{pmatrix}$ row-reduces to $B = \begin{pmatrix} 1 & 0 & 0 & -4/11 \\ 0 & 1 & 0 & 13/11 \\ 0 & 0 & 1 & 10/11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

a)Find all solutions to $A\mathbf{x} = 0$.

These are the same as the solutions to $B\mathbf{x} = 0$, namely all multiples of $(4/11, -13/11, -10/11, 1)^T$, or equivalently all multiples of $(4, -13, -10, 11)^T$.

b)Find a basis for the column space of A.

Since there are pivots in the first three columns of B, the first three columns of A for a basis. That is, the answer is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 5 \end{pmatrix} \right\}$.

c)In $\mathbf{R}_3[t]$, let V be the span of the vectors $\{1 + 2t + 3t^3, 3 - t + t^2 + 3t^3, 2 + t + 2t^2 + 5t^3, 5 - t + 3t^2 + 7t^3\}$. What is the dimension of V? Find a basis for V.

If you express things in coordinates with respect to the standard basis $\{1, t, t^2, t^3\}$, this becomes the same problem as (b). V is 3-dimensional, as a basis consists of those polynomials whose coordinates are the answer to (b), namely $\{1+2t+3t^3,\ 3-t+t^2+3t^3,\ 2+t+2t+2+5t^3\}$. Note that the answer is NOT a matrix or a list of column vectors. Those are just the *coordinates* of the answer, not the answer itself.

2. a) Find the eigenvalues of $\begin{pmatrix} 3 & -5 & 16 & 4 \\ 0 & 3 & 11 & 0 \\ 0 & 15 & -1 & 0 \\ 0 & 4 & 1 & 2 \end{pmatrix}$. You do not need to find the eigenvectors.

The matrix is block triangular, with an upper left 1×1 block and a lower right 3×3 block. The 3×3 block is itself block triangular, with an upper left 2×2 piece $\begin{pmatrix} 3 & 11 \\ 15 & -1 \end{pmatrix}$ and a lower right 1×1 piece. The rows of the 2×2 piece sum to 14, and the trace is 2, so that piece has eigenvalues 14 and -12, and the whole matrix has eigenvalues $3, 14, -12, 2$.

b) Find the eigenvalues *and eigenvectors* of $\begin{pmatrix} 3 & 8 \\ 2 & -3 \end{pmatrix}$.

The determinant is -25 and the trace is zero, so the eigenvalues are 5 and -5. The eigenvectors (obtained by row reductions) are $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, respectively.

3. Consider the equations

$$\begin{aligned} x_1(n+1) &= 2x_1(n) + 3x_2(n) \\ x_2(n+1) &= 2x_1(n) + x_2(n) \end{aligned}$$

a) If $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, what is $\mathbf{x}(n)$?

Since this is an $\mathbf{x}(n+1) = A\mathbf{x}(n)$ problem, we diagonalize A and get eigenvalues 4 and -1 with eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Since $\mathbf{x}(0) = (\mathbf{b}_1 + 2\mathbf{b}_2)/5$, $\mathbf{x}(n) = (4^n\mathbf{b}_1 + 2(-1)^n\mathbf{b}_2)/5 = \frac{1}{5}\begin{pmatrix} 3 \cdot 4^n + 2(-1)^n \\ 2 \cdot 4^n - 2(-1)^n \end{pmatrix}$.

b) If $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, what is $\mathbf{x}(n)$?

In this instance $\mathbf{x}(0) = (\mathbf{b}_1 - 3\mathbf{b}_2)/5$, so $\mathbf{x}(n) = (4^n\mathbf{b}_1 - 3(-1)^n\mathbf{b}_2)/5 = \frac{1}{5}\begin{pmatrix} 3 \cdot 4^n - 3(-1)^n \\ 2 \cdot 4^n + 3(-1)^n \end{pmatrix}$.

c) Compute A^n, where $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.

You can either compute PD^nP^{-1} or notice that the first column of A^n is the answer to (a) and the second column is the answer to (b). Either way, you get $A^n = \frac{1}{5}\begin{pmatrix} 3 \cdot 4^n + 2(-1)^n & 3 \cdot 4^n - 3(-1)^n \\ 2 \cdot 4^n - 2(-1)^n & 2 \cdot 4^n + 3(-1)^n \end{pmatrix}$.

4. Consider the nonlinear system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_1^2 + x_1x_2 - 4x_1 + x_2 + 1 \\ \frac{dx_2}{dt} &= x_2^2 + x_1 - 2x_2 \end{aligned}$$

This system of equations has a fixed point at $x_1 = x_2 = 1$.

a) Write down a linear system of equations that approximates this nonlinear system when $\mathbf{x}$ is close to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Defining $\mathbf{y} = \mathbf{x} - (1,1)^T$, we get $\frac{d\mathbf{y}}{dt} \approx A\mathbf{y}$, where

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\Bigg|_{(1,1)} = \begin{pmatrix} 2x_1 + x_2 - 4 & x_1 + 1 \\ 1 & 2x_2 - 2 \end{pmatrix}\Bigg|_{(1,1)} = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$$

b) Diagonalize the matrix that appears in the linear equations.

Eigenvalues -2 and 1, with eigenvectors $(-2,1)^T$ and $(1,1)^T$.

c) Identify the stable, neutrally stable, and unstable modes. What is the dominant mode, and how fast does it grow or shrink? Is the system as a whole stable, neutral, or unstable near $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

The $(-2,1)^T$ mode is stable, and shrinks as e^{-2t}. The $(1,1)^T$ mode is unstable, and grows as e^t. That's the dominant mode. Since there is an unstable mode, the system is unstable.

5. Gram-Schmidt. In $\mathbf{R}^3$, consider the three vectors $\mathbf{x}_1 = (2,1,1)^T$, $\mathbf{x}_2 = (5,-1,3)^T$ and $\mathbf{x}_3 = (4,6,-8)^T$.

a) Use Gram-Schmidt to convert this basis to an orthogonal basis $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

$\mathbf{y}_1 = \mathbf{x}_1 = (2,1,1)^T$.

$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{y}_1 | \mathbf{x}_2 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle}\mathbf{y}_1 = \mathbf{x}_2 - (12/6)\mathbf{y}_1 = (1,-3,1)^T$.

$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{y}_1 | \mathbf{x}_3 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle}\mathbf{y}_1 - \frac{\langle \mathbf{y}_2 | \mathbf{x}_3 \rangle}{\langle \mathbf{y}_2 | \mathbf{y}_2 \rangle}\mathbf{y}_2 = \mathbf{x}_3 - \mathbf{y}_1 + 2\mathbf{y}_2 = (4,-1,-7)^T$.

b) Decompose the vector $(1,2,3)^T$ as a linear combination of the vectors in this orthogonal basis. (Warning: the answer involves fractions.)

Let $\mathbf{v} = (1,2,3)^T$. Since $\langle \mathbf{y}_1 | \mathbf{v} \rangle = 7$, $\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle = 6$, $\langle \mathbf{y}_2 | \mathbf{v} \rangle = -2$, $\langle \mathbf{y}_2 | \mathbf{y}_2 \rangle = 11$, $\langle \mathbf{y}_3 | \mathbf{v} \rangle = -19$, $\langle \mathbf{y}_3 | \mathbf{y}_3 \rangle = 66$, $\mathbf{v} = \frac{6}{7}\mathbf{y}_1 - \frac{2}{11}\mathbf{y}_2 - \frac{19}{66}\mathbf{y}_3$.

6. Let V be the space of functions on the interval $[0,\pi]$ with boundary conditions $f(0) = 0$, $f(\pi) = 0$.

a) Let $A = 4 + \frac{d^2}{dx^2}$ be an operator on V. (In other words, $(Af)(x) = 4f(x) + f''(x)$) Find all the eigenvalues and eigenvectors of A.

We already know that the eigenvalues of d^2/dx^2 are $-n^2\pi^2/L^2 = -n^2$ with eigenvectors $\sin(n\pi x/L) = \sin(nx)$, so the eigenvalues of A are $4 - n^2$ with the same eigenvectors (or, if you prefer, eigenfunctions) $\sin(nx)$.

b) Consider the partial differential equation

$$\frac{\partial^2 f(x,t)}{\partial t^2} = 4f(x,t) + \frac{\partial^2 f(x,t)}{\partial x^2}$$

on $[0,\pi] \times \mathbf{R}$, and with the boundary conditions $f(0,t) = f(\pi,t) = 0$ for all t. Find a solution to this equation with the initial conditions $f(x,0) = \sin(x) - 5\sin(3x)$, $\frac{\partial f}{\partial t}(x,0) = 3\sin(2x)$.

This is of the form $\frac{d^2\vec{f}}{dt^2} = A\vec{f}$, so we break things down in a basis of eigenvectors of A. The modes with $\lambda > 0$ grow as $\cosh(\sqrt{\lambda}t)$ and $\sinh(\sqrt{\lambda}t)$, the modes with $\lambda = 0$ go as $c_1 + c_2 t$, and the modes with $\lambda < 0$ go as $\cos(\sqrt{-\lambda}t)$ and $\sin(\sqrt{-\lambda}t)$.

Our initial conditions only have elements in the $n=1$, $n=2$ and $n=3$ eigenspaces, which have positive, zero, and negative eigenvalues, and our final answer is

$$f(x,t) = \cosh(\sqrt{3}t)\sin(x) + 3t\sin(2x) - 5\cos(\sqrt{5}t)\sin(3x).$$

7. Consider the "sawtooth function", defined by $f(x) = x$ for $0 < x < 1$ and with $f(x+1) = f(x)$. (This function is discontinuous when x is an integer.)
a) We write $f(x) = \sum_n \hat{f}_n \exp(2\pi i n x)$ as a Fourier series. Find the Fourier coefficients $\hat{f}_n$.

$\hat{f}_n = \int_0^1 x e^{-2\pi i n x} dx$. For $n = 0$ this equals 1/2. For any other value of n we integrate by parts, using the fact that $\int x e^{kx} dx = \frac{xe^{kx}}{k} - \int \frac{e^{kx}}{k} dx = \frac{(kx-1)e^{kx}}{k^2}$. Plugging in $k = 2\pi i n$ and noting that $e^{2\pi i n x}$ equals 1 at $x = 0$ and $x = 1$, we get $\hat{f}_n = \frac{i}{2\pi n}$ for $n \neq 0$.
b) We can also write $f(x)$ as a sum of sines and cosines: $f(x) = \frac{a_0}{2} + \sum_n a_n \cos(2\pi n x) + \sum_n b_n \sin(2\pi n x)$. Find the coefficients a_n and b_n.

$a_n = \hat{f}_n + \hat{f}_{-n}$. This is 1 if $n = 0$ and 0 otherwise.

$b_n = i\hat{f}_n - i\hat{f}_{-n} = -\frac{1}{n\pi}$.
c) Suppose that $g(x)$ is a periodic function that solves the equation $d^2g(x)/dx^2 = f(x) - \frac{1}{2}$. Find the Fourier coefficients $\hat{g}_n$ for all $n \neq 0$. ($\hat{g}_0$ is a constant of integration and is arbitrary.)

Since $\hat{g''}_n = -4n^2\pi^2\hat{g}_n$, we have that $-4n^2\pi^2\hat{g}_n = \hat{f}_n = i/2\pi n$, so $\hat{g}_n = \frac{-i}{8\pi^3 n^3}$.

8. True or false? (2 points each, no partial credit, and no penalty for guessing.)

a) Every standing wave on the interval $[0, L]$, with Dirichlet boundary conditions, can be written as a sum of traveling waves.

True. One can use either standing or traveling waves.

b) If R is a rotation in 3-dimensional space, then the trace of R is at least -1.

True. The trace is $1 + 2\cos(\theta)$, which can be anything from -1 to 3.

c) If B is a complex anti-symmetric matrix ($B^T = -B$), then e^B is unitary.

False. For instance, the exponential of $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ has eigenvalues e and e^{-1}, and is definitely not unitary.

d) If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of a Hermitian matrix A, then $\langle \mathbf{x}|\mathbf{y}\rangle = 0$.

False. If they have the same eigenvalue, they don't have to be orthogonal.

e) Suppose that A is a 5×5 matrix with determinant 0 and trace 5. If 1 is an eigenvalue with geometric multiplicity 3 then A is diagonalizable.

True. Since the determinant is 0, one of the eigenvalues has to be 0. From the trace, we see that the last eigenvalue is 2. Since 1 has both geometric and algebraic multiplicity 3, and the others have geometric and algebraic multiplicity 1, the matrix is diagonalizable.

f) If A is a 3×5 matrix and $\mathbf{b} \in \mathbf{R}^3$, then there are infinitely many solutions to $A\mathbf{x} = \mathbf{b}$.

False. There may not be any solutions. (But if there are any solutions, there are infinitely many.)

g) If A is an $m \times n$ matrix and $\mathbf{b} \in \mathbf{R}^m$, then there exists a least-squares solution to $A\mathbf{x} = \mathbf{b}$, no matter what A and $\mathbf{b}$ are.

True. Least squares solutions always exist.

h) If $\mathcal{B}$ and $\mathcal{D}$ are different bases for a vector space V and $L : V \to V$ is an operator, then $[L]_{\mathcal{B}}$ and $[L]_{\mathcal{D}}$ have the same eigenvalues.

True, and the eigenvectors are related by the change-of-basis matrix.