M346 First Midterm Exam Solutions, September 18, 2009

1) In $\mathbb{R}^2$, let $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis and let $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \end{pmatrix} \right\}$ be an alternate basis.

a) Find $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$.

$P_{\mathcal{E}\mathcal{B}} = ([\mathbf{b}_1]_{\mathcal{E}}[bb_2]_{\mathcal{E}}) = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$. $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} -7 & 5 \\ 3 & -2 \end{pmatrix}$. *Here we used the fact that* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

b) If $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, find $[\mathbf{v}]_{\mathcal{B}}$.

Since $[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $[\mathbf{v}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} -23 \\ 10 \end{pmatrix}$.

c) Solve the system of equations: $2x_1 + 5x_2 = 4; \quad 3x_1 + 7x_2 = 1$.

This is the exact same problem as (b), namely writing $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ *as a linear combination of* $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *and* $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$. *The solution, as before, is* $x_1 = -23$, $x_2 = 10$.

2. In $\mathbb{R}_1[t]$, let $\mathcal{E} = \{1, t\}$ be the standard basis and let $\mathcal{B} = \{4 + 5t, 3 + 4t\}$ be an alternate basis. Let $L : \mathbb{R}_1[t] \to \mathbb{R}_1[t]$ be the linear transformation $L(a_0 + a_1 t) = (16a_0 - 12a_1) + (20a_0 - 15a_1)t$.

a) Find $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$.

$P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$, $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix}$.

b) Find the matrix of L relative to the standard basis (that is, find $[L]_{\mathcal{E}}$).

$[L]_{\mathcal{E}} = ([L(\mathbf{e}_1)]_{\mathcal{E}}[L(\mathbf{e}_2)]_{\mathcal{E}}) = \begin{pmatrix} 16 & -12 \\ 20 & -15 \end{pmatrix})$

c) Find the matrix of L relative to the basis $\mathcal{B}$ (that is, find $[L]_{\mathcal{B}}$). [The answer to (c) is much simpler than the answer to (b) and illustrates why we use bases like $\mathcal{B}$.]

There are two ways to do this. We could compute $L(\mathbf{b}_1) = 4 + 5t = \mathbf{b}_1$, *hence* $[L(\mathbf{b}_1)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, *and compute* $L(\mathbf{b}_2) = 0$ *to conclude that* $[L]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, *or we could compute* $P_{\mathcal{B}\mathcal{E}}[L]_{\mathcal{E}}P_{\mathcal{E}\mathcal{B}}$ *to get the same result.*

3. Let $A = \begin{pmatrix} 1 & 2 & 5 & 1 & 15 \\ 2 & 0 & 6 & 1 & 10 \\ 3 & 2 & 11 & 1 & 13 \\ 6 & 4 & 22 & 3 & 38 \end{pmatrix}$. A is row-equivalent to $A_{rref} = \begin{pmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

a) Find a basis for the space of solutions to $A\mathbf{x} = 0$.

Our equations are $x_1 = -3x_3 + x_5$, $x_2 = -x_3 - 2x_5$, $x_4 = -12x_5$, *and of course* $x_3 = x_3$ *and* $x_5 = x_5$. *In other words* $\mathbf{x} = x_3\mathbf{b}_1 + x_5\mathbf{b}_2$, *where* $\mathbf{b}_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ *and* $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ -12 \\ 1 \end{pmatrix}$ *form a basis for our space of solutions.*

b) Find a basis for the column space of A.

These are the first, second, and fourth columns of A, *corresponding to the pivot columns of* A_{rref}. *That is,* $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 2 \\ 4 \end{pmatrix}$, *and* $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$. *There are other possible bases, of course, but this is the one described in class and in Appendix A.*

c) Let L:$\mathbb{R}_4[t] \to \mathbb{R}_3[t]$ be a linear transformation with $[L]_{\tilde{\mathcal{E}}\mathcal{E}} = A$. Here $\mathcal{E} = \{1, t, t^2, t^3, t^4\}$ is the standard basis for $\mathbb{R}_4[t]$ and $\tilde{\mathcal{E}} = \{1, t, t^2, t^3\}$ is the standard basis for $\mathbb{R}_3[t]$. Find bases for Ker(L) (the kernel of L) and Range(L).

This is essentially the same as parts (a) and (b)! A basis for the kernel of L is given by the vectors whose coordinates are the answer to (a), namely $\{-3 - t + t^2, 1 - 2t - 12t^3 + t^4\}$, *and a basis for the range are the vectors whose coordinates are the answer to (b), namely* $\{1 + 2t + 3t^2 + 6t^3, 2 + 2t^2 + 4t^3, 1 + t + t^2 + 3t^3\}$.

4. True of False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) If A is a matrix, then the pivot columns of A_{rref} form a basis for the column space of A.

False. A basis if formed from the columns of A, *not the columns of* A_{rref}.

b) The set of solutions to $A\mathbf{x} = 0$ is the same as the set of solutions to $A_{rref}\mathbf{x} = 0$.

True

c) If $L : V \to V$ is an operator and $\mathcal{B}$ and $\mathcal{D}$ are bases for V, then $[L]_{\mathcal{B}} = P_{\mathcal{DB}}[L]_{\mathcal{D}}P_{\mathcal{BD}}$.

False. You need to switch $P_{\mathcal{DB}}$ and $P_{\mathcal{BD}}$ to get the right formula.

d) If V is an n-dimensional vector space with a basis $\mathcal{B}$, then a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in V is linearly independent if and only if the matrix $A = ([\mathbf{v}_1]_{\mathcal{B}} \ldots [\mathbf{v}_k]_{\mathcal{B}})$ has rank k.

True. You need a pivot in each of the k columns.

e) If V is an n-dimensional vector space with a basis $\mathcal{B}$, then a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ spans V if and only if the matrix $A = ([\mathbf{v}_1]_{\mathcal{B}} \ldots [\mathbf{v}_k]_{\mathcal{B}})$ has rank n.

True. You need a pivot in each of the n rows.