M346 First Midterm Exam, September 18, 2009

1) In $\mathbb{R}^2$, let $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis and let $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \end{pmatrix} \right\}$ be an alternate basis.

a) Find $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$.

b) If $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, find $[\mathbf{v}]_{\mathcal{B}}$.

c) Solve the system of equations: $2x_1 + 5x_2 = 4; \quad 3x_1 + 7x_2 = 1$.

2. In $\mathbb{R}_1[t]$, let $\mathcal{E} = \{1, t\}$ be the standard basis and let $\mathcal{B} = \{4 + 5t, 3 + 4t\}$ be an alternate basis. Let $L : \mathbb{R}_1[t] \to \mathbb{R}_1[t]$ be the linear transformation $L(a_0 + a_1 t) = (16a_0 - 12a_1) + (20a_0 - 15a_1)t$.

a) Find $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$.

b) Find the matrix of L relative to the standard basis (that is, find $[L]_{\mathcal{E}}$).

c) Find the matrix of L relative to the basis $\mathcal{B}$ (that is, find $[L]_{\mathcal{B}}$).

[The answer to (c) is much simpler than the answer to (b) and illustrates why we use bases like $\mathcal{B}$.]

3. Let $A = \begin{pmatrix} 1 & 2 & 5 & 1 & 15 \\ 2 & 0 & 6 & 1 & 10 \\ 3 & 2 & 11 & 1 & 13 \\ 6 & 4 & 22 & 3 & 38 \end{pmatrix}$. A is row-equivalent to $A_{rref} = \begin{pmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

a) Find a basis for the space of solutions to $A\mathbf{x} = 0$.

b) Find a basis for the column space of A.

c) Let $L : \mathbb{R}_4[t] \to \mathbb{R}_3[t]$ be a linear transformation with $[L]_{\tilde{\mathcal{E}}\mathcal{E}} = A$. Here $\mathcal{E} = \{1, t, t^2, t^3, t^4\}$ is the standard basis for $\mathbb{R}_4[t]$ and $\tilde{\mathcal{E}} = \{1, t, t^2, t^3\}$ is the standard basis for $\mathbb{R}_3[t]$. Find bases for Ker(L) (the kernel of L) and Range(L).

4. True of False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) If A is a matrix, then the pivot columns of A_{rref} form a basis for the column space of A.

b) The set of solutions to $A\mathbf{x} = 0$ is the same as the set of solutions to $A_{rref}\mathbf{x} = 0$.

c) If $L : V \to V$ is an operator and $\mathcal{B}$ and $\mathcal{D}$ are bases for V, then $[L]_{\mathcal{B}} = P_{\mathcal{D}\mathcal{B}}[L]_{\mathcal{D}}P_{\mathcal{B}\mathcal{D}}$.

d) If V is an n-dimensional vector space with a basis $\mathcal{B}$, then a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in V is linearly independent if and only if the matrix $A = ([\mathbf{v}_1]_{\mathcal{B}} \ldots [\mathbf{v}_k]_{\mathcal{B}})$ has rank k.

e) If V is an n-dimensional vector space with a basis $\mathcal{B}$, then a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ spans V if and only if the matrix $A = ([\mathbf{v}_1]_{\mathcal{B}} \ldots [\mathbf{v}_k]_{\mathcal{B}})$ has rank n.