M346 Third Midterm Exam, April 24, 2009

1. **Is it fixed yet?** Consider the system of nonlinear differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(3 - 2x_1 - x_2) \\ \frac{dx_2}{dt} &= x_2(5 - 2x_1 - 3x_2). \end{aligned}$$

a) Find all the fixed points.

b) For each fixed point, indicate how many stable modes, and how many unstable modes, there are.

2. **Gram crackers.** In $\mathbb{R}^3$, with the standard inner product, consider the vectors $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 16 \\ 9 \\ -2 \end{pmatrix}$ that form a basis for $\mathbb{R}^3$.

a) Use the Gram-Schmidt process to convert this basis to an orthogonal basis $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$. (Note: the vectors $\mathbf{y}_i$ do not have to be orthonormal, just orthogonal.)

b) The vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ can be expressed as $c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3$. Find c_1, c_2 and c_3.

3. **When least is best.** Find **all** least-squares solutions to the system of equations

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 1 \\ 3x_1 + 6x_2 &= 1 \\ 4x_1 + 8x_2 &= 1 \end{aligned}$$

4. **Working 24/7.** Consider the Hermitian matrix $H = \begin{pmatrix} 24 & 7 \\ 7 & -24 \end{pmatrix}$.

a) Find the eigenvalues λ_1 and λ_2 and corresponding eigenvectors $\mathbf{b}_1$ and $\mathbf{b}_2$ of H.

b) Decompose $\mathbf{x}_0 = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$ as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$.

c) If $d\mathbf{x}/dt = H\mathbf{x}$ and $\mathbf{x}(0) = \mathbf{x}_0$ as in (b), what is $\mathbf{x}(t)$?