1) In \( \mathbb{R}^2 \), consider the operator \( L(x) = Ax \), where \( A = \begin{pmatrix} 5 & 10 \\ -15 & 20 \end{pmatrix} \). Consider the basis \( \mathcal{B} = \{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \} \) and the vector \( x = \begin{pmatrix} 120 \\ 70 \end{pmatrix} \).

a) Find the coordinates of \( x \) in the \( \mathcal{B} \) basis. (That is, find \([x]_{\mathcal{B}}\).)

\[
P_{\mathcal{B}\mathcal{E}} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, 
\text{so} \quad P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}, \quad \text{so} \quad [x]_{\mathcal{B}} = P_{\mathcal{E}\mathcal{B}}x = \begin{pmatrix} 58 \\ 4 \end{pmatrix}. \quad \text{You can check that} \quad \begin{pmatrix} 120 \\ 70 \end{pmatrix} \text{is indeed} \ 58 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 3 \end{pmatrix}.
\]

b) Find the coordinates of \( L \) in the \( \mathcal{B} \) basis, that is \([L]_{\mathcal{B}}\).

\[
[L]_{\mathcal{B}} = P_{\mathcal{E}\mathcal{B}}AP_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 14 & 12 \\ -8 & 11 \end{pmatrix}.
\]

2. Let \( V = \mathbb{R}_2[t] \), the space of quadratic polynomials in a variable \( t \). On \( V \), consider the operator \( (L(p))(t) = p(2t + 1) \), where the right hand side means the polynomial \( p \) evaluated at the point \( 2t + 1 \). (If \( p(t) \) were the function \( \sin(t) \), then \( L(p) \) would be the function \( \sin(2t + 1) \). Of course, \( p \) is a polynomial rather than a trig function, but the rule for how \( L \) acts is the same.)

a) Find the matrix of \( L \) with respect to the basis \( \mathcal{E} = \{ 1, t, t^2 \} \).

\[
L(1) = 1, \quad L(t) = 1 + 2t, \quad \text{and} \quad L(t^2) = (2t + 1)^2 = 1 + 4t + 4t^2, \quad \text{so the matrix is} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}.
\]

b) Find all solutions to \( L(p) = 2p \).

This is another way of saying “find all eigenvectors of \( L \) with eigenvalue \( 2 \)”. These are the vectors in \( V \) whose coordinates are eigenvectors of \([L]_{\mathcal{B}}\) with eigenvalue \( 2 \). By row reduction, you get that \([p]_{\mathcal{B}}\) must be a multiple of \( \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \), so \( p \) itself must be a multiple of \( 1 + t \).

3. Diagonalization.

a) Find the characteristic polynomial of the matrix \( A = \begin{pmatrix} 1 & -2 & -3 \\ -3 & 0 & -2 \\ -4 & -1 & 0 \end{pmatrix} \).
You do not need to find the eigenvalues or eigenvectors.

\[ p_A(\lambda) = \det \begin{pmatrix} \lambda - 1 & 2 & 3 \\ 3 & \lambda & 2 \\ 4 & 1 & \lambda \end{pmatrix} = \lambda^3 - \lambda^2 - 20\lambda + 27. \] No, I don’t expect you to find the roots of this!

Note that the definition of \( p_A(\lambda) \) is \( \det(\lambda I - A) \), not \( \det(A - \lambda I) \)!

These expressions differ by a factor of \((-1)^n\).

b) Find the eigenvalues of \( B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -3 & 6 & 0 & 0 \\ 3 & 5 & 2 & 3 \\ 2 & 9 & -3 & 2 \end{pmatrix} \). You do not need to find the eigenvectors.

This is block triangular, so we just need to find the eigenvalues of \( \begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix} \) and the eigenvalues of \( \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \). These can be done by finding the characteristic polynomial and finding its roots, but there are easier ways. For the first matrix, the rows all sum to 3 and the trace is 8, so the eigenvalues must be 3 and 5. For the second matrix, this is of the form \( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \) with eigenvalues \( a \pm bi \), so our eigenvalues are \( 2 - 3i \) and \( 2 + 3i \). Put everything together, and the eigenvalues are 3, 5, and \( 2 \pm 3i \).

c) Find the eigenvalues and eigenvectors of \( C = \begin{pmatrix} 5 & 2 \\ -1 & 2 \end{pmatrix} \).

The trace is 7 and the determinant is 12, so the eigenvalues are 3 and 4. (Or you can compute the characteristic polynomial and apply the quadratic formula.) By row reduction, the eigenvectors are found to be \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

4. Consider the matrix \( A = \frac{1}{5} \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix} \).

a) Is the system of equations \( x(n+1) = Ax(n) \) stable or unstable? What is/are the dominant eigenvalue(s)?

The eigenvalues of \( A \) are \(-4/5\) and \((1 \pm 2i)/5\), with corresponding eigen-
vectors $b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $b_{2,3} = \begin{pmatrix} 0 \\ \pm i \\ 1 \end{pmatrix}$. All of these have magnitude less than 1, so the system is stable. The biggest eigenvalue (in magnitude) is $-4/5$, so that’s the dominant eigenvalue.

b) Find a solution to $x(n+1) = Ax(n)$ with initial condition $x(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(You can leave your answer as a linear combination of eigenvectors.)

$$x(0) = b_1 - \frac{i}{2}b_2 + \frac{i}{2}b_3,$$

so $x(n) = (-4/5)^n b_1 - \frac{i}{2} \left(\frac{1+2i}{5}\right)^n b_2 + \frac{i}{2} \left(\frac{1-2i}{5}\right)^n b_3$.

c) Now consider the system of differential equations $\frac{dx}{dt} = Ax$. Is the system stable, neutral, or unstable? What is/are the dominant eigenvalues?

Since $(1 \pm 2i)/5$ have positive real part, the system is unstable. The dominant eigenvalue is the one with the greatest real part. This is a tie between $(1 \pm 2i)/5$.

d) Find a solution to $\frac{dx}{dt} = Ax$ with initial condition $x(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$$x(t) = e^{-4t/5} b_1 - \frac{i}{2} e^{\left(\frac{1+2i}{5}\right)t} b_2 + \frac{i}{2} e^{\left(\frac{1-2i}{5}\right)t} b_3.$$ This has the same form as the solution to part (b), only with $e^{\lambda t}$ instead of $\lambda^n$. If you add up the different terms, it actually simplifies greatly, to $x(t) = \begin{pmatrix} e^{-4t/5} e^t \cos(2t) \\ e^{-4t/5} e^t \sin(2t) \end{pmatrix}$.

5. Orthogonality. In $\mathbb{R}^3$, let $V$ be the span of the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$.

This problem is essentially Gram-Schmidt applied to the three vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$, and $\begin{pmatrix} 70 \\ 0 \\ 0 \end{pmatrix}$, except that I asked about the first two vectors in part (a) and the third vector in part (b).

a) Use Gram-Schmidt to find an orthogonal basis for $V$.

$$y_1 = x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$
\[ y_2 = x_2 - \frac{(y_1|x_2)}{(y_1|y_1)}y_1 = x_2 - \frac{28}{14}y_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}. \]

b) Let \( x = \begin{pmatrix} 70 \\ 0 \\ 0 \end{pmatrix} \). Write \( x \) as the sum of two vectors, one in \( V \) and the other orthogonal to \( V \).

We want to write \( x = v + w \), where \( v \in V \) and \( w \perp V \). \( v \) is the projection of \( x \) onto \( V \), and equals \( \frac{70}{14}y_1 + \frac{-210}{10}y_2 = \begin{pmatrix} 68 \\ 10 \\ -6 \end{pmatrix} \). \( w \) is what’s left over, namely \( \begin{pmatrix} 2 \\ -10 \\ 6 \end{pmatrix} \).

6. a) On \( \mathbb{C}^3 \), let the operator \( L \) be given by the rule \( L(x) = \begin{pmatrix} 3x_1 + 5x_2 - x_3 \\ 4x_1 + ix_2 + x_3 \\ 7x_1 - x_2 + ix_3 \end{pmatrix} \).

Compute \( L^\dagger(x) \).

The matrix of \( L \) is \( \begin{pmatrix} 3 & 5 & -1 \\ 4 & i & 1 \\ 7 & -1 & i \end{pmatrix} \), so the matrix of \( L^\dagger \) is \( \begin{pmatrix} 3 & 4 & 7 \\ 5 & -i & -1 \\ -1 & 1 & -i \end{pmatrix} \),

so \( L^\dagger(x) = \begin{pmatrix} 3x_1 + 4x_2 + 7x_3 \\ 5x_1 - ix_2 - x_3 \\ -x_1 + x_2 - ix_3 \end{pmatrix} \).

b) Let \( A = \begin{pmatrix} 0 & -3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \), let \( B = e^A \), and let \( C = e^{\pi A} \). Which of these matrices are Hermitian? Which are anti-hermitian? Which are orthogonal? Explain.

\( A \) is real anti-symmetric, hence anti-hermitian, which means that the eigenvalues are pure imaginary (\( \pm 3i \) and \( \pm 2i \)) and the eigenvectors are orthogonal. The eigenvectors of \( A \) are also the eigenvectors of \( B \) and \( C \). The eigenvalues of \( B \) are \( e^{\pm 2i}, e^{\pm 3i} \) which are complex and on the unit circle, while the eigenvalues of \( C \) are \( e^{\pm 2\pi i} = 1 \) and \( e^{\pm 3\pi i} = -1 \). These are both on the unit circle and are real. Since \( A \) is a real matrix, both \( B \) and \( C \) are real.
matrices. Thus, $C$ is Hermitian, $A$ is anti-Hermitian, and both $B$ and $C$ are orthogonal.

7. Working on the interval $[0, 1]$, let $f_0(x) = 1$ for $0 < x < 1$. We write this function as a Fourier series $f_0(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$.

a) Compute the coefficients $a_n$.

$$a_n = \frac{2}{1} \int_{0}^{1} 1 \times \sin(n\pi x) \, dx = \frac{2}{n\pi} \cos(n\pi) \big|_{0}^{1}.$$

This equals $\frac{4}{n\pi}$ if $n$ is odd and 0 if $n$ is even.

b) Now suppose that $f(x, t)$ satisfies the “heat equation”

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2},$$

with Dirichlet boundary conditions $f(0, t) = f(1, t) = 0$. [Physical note: $f(x, t)$ describes the temperature of a point $x$ along a rod of length 1 at time $t$, where the ends of the rod are in contact with heat sinks at temperature 0.] Viewing this as an ordinary differential equation ($\frac{df}{dt} = L(f)$) in a space of functions, what is the dominant mode? Is it stable or unstable? How quickly does it grow or shrink?

This is analogous to solving the wave equation in terms of standing waves, except that our equation is $\frac{df}{dt} = Af$ rather than $\frac{d^2 f}{dt^2} = Af$, where $A$ is the operator $\frac{d^2}{dx^2}$. As with the standing wave calculation, the eigenvalues of $A$ are $-n^2\pi^2$, and the eigenvectors are $\sin(n\pi x)$. All eigenvalues are negative, meaning that all terms shrink away with time, but the least negative eigenvalue (i.e., the dominant eigenvalue) is $-\pi^2$, meaning that the term that survives the longest goes as $e^{-\pi^2 t}$.

c) Find the solution $f(x, t)$ for all (non-negative) $t$, starting with initial condition $f(x, 0) = f_0(x)$. You can leave your answer as a series. [Note: The initial condition is discontinuous at $x = 0$ and $x = 1$, since the entire rod is hot but the surroundings are cold, but the solution quickly becomes smooth.]

$$f(x, t) = \sum_{n \text{ odd}} a_n e^{\lambda_n t} \sin(n\pi x) = \sum_{n \text{ odd}} \frac{4}{n\pi} e^{-n^2\pi^2 t} \sin(n\pi x).$$