M346 Final Exam, May 14, 2011

1) In $\mathbb{R}^2$, consider the operator $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{pmatrix} 5 & 10 \\ -15 & 20 \end{pmatrix}$. Consider the basis $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ and the vector $\mathbf{x} = \begin{pmatrix} 120 \\ 70 \end{pmatrix}$.

a) Find the coordinates of $\mathbf{x}$ in the $\mathcal{B}$ basis. (That is, find $[\mathbf{x}]_{\mathcal{B}}$.)

b)Find the coordinates of L in the $\mathcal{B}$ basis, that is $[L]_{\mathcal{B}}$.

2. Let $V = \mathbb{R}_2[t]$, the space of quadratic polynomials in a variable t. On V, consider the operator $(L(\mathbf{p}))(t) = \mathbf{p}(2t+1)$, where the right hand side means the polynomial $\mathbf{p}$ evaluated at the point $2t+1$. (If $\mathbf{p}(t)$ were the function $sin(t)$, then $L(\mathbf{p})$ would be the function $\sin(2t+1)$. Of course, $\mathbf{p}$ is a polynomial rather than a trig function, but the rule for how L acts is the same.)

a) Find the matrix of L with respect to the basis $\mathcal{E} = \{1, t, t^2\}$.

b) Find all solutions to $L(\mathbf{p}) = 2\mathbf{p}$.

3. Diagonalization.

a) Find the characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & -2 & -3 \\ -3 & 0 & -2 \\ -4 & -1 & 0 \end{pmatrix}$.

You do not need to find the eigenvalues or eigenvectors.

b) Find the eigenvalues of $B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -3 & 6 & 0 & 0 \\ 3 & 5 & 2 & 3 \\ 2 & 9 & -3 & 2 \end{pmatrix}$. You do not need to find the eigenvectors.

c) Find the eigenvalues and eigenvectors of $C = \begin{pmatrix} 5 & 2 \\ -1 & 2 \end{pmatrix}$.

4. Consider the matrix $A = \frac{1}{5}\begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$.

a) Is the system of equations $\mathbf{x}(n+1) = A\mathbf{x}(n)$ stable or unstable? What is/are the dominant eigenvalue(s)?

b) Find a solution to $\mathbf{x}(n+1) = A\mathbf{x}(n)$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(You can leave your answer as a linear combination of eigenvectors.)
c) Now consider the system of differential equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. Is the system stable, neutral, or unstable? What is/are the dominant eigenvalues?

d) Find a solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

5. Orthogonality. In $\mathbb{R}^3$, let V be the span of the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$.

a) Use Gram-Schmidt to find an orthogonal basis for V.

b) Let $\mathbf{x} = \begin{pmatrix} 70 \\ 0 \\ 0 \end{pmatrix}$. Write $\mathbf{x}$ as the sum of two vectors, one in V and the other orthogonal to V.

6. a) On $\mathbb{C}^3$, let the operator L be given by the rule $L(\mathbf{x}) = \begin{pmatrix} 3x_1 + 5x_2 - x_3 \\ 4x_1 + ix_2 + x_3 \\ 7x_1 - x_2 + ix_3 \end{pmatrix}$.

Compute $L^\dagger(\mathbf{x})$.

b) Let $A = \begin{pmatrix} 0 & -3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$, let $B = e^A$, and let $C = e^{\pi A}$. Which of these matrices are Hermitian? Which are anti-hermitian? Which are orthogonal? Explain.

7. Working on the interval $[0, 1]$, let $f_0(x) = 1$ for $0 < x < 1$. We write this function as a Fourier series $f_0(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$.

a) Compute the coefficients a_n.

b) Now suppose that $f(x, t)$ satisfies the "heat equation"

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2},$$

with Dirichlet boundary conditions $f(0, t) = f(1, t) = 0$. [Physical note: $f(x, t)$ describes the temperature of a point x along a rod of length 1 at time t, where the ends of the rod are in contact with heat sinks at temperature 0.] Viewing this as an ordinary differential equation ($d\mathbf{f}/dt = L(\mathbf{f})$) in a space of functions, what is the dominant mode? Is it stable or unstable? How quickly does it grow or shrink?

c) Find the solution $f(x, t)$ for all (non-negative) t, starting with initial condition $f(x, 0) = f_0(x)$. You can leave your answer as a series. [Note: The initial condition is discontinuous at $x = 0$ and $x = 1$, since the entire rod is hot but the surroundings are cold, but the solution quickly becomes smooth.]