1) (15 points) Consider the vectors \(
\begin{pmatrix}
1 \\
1 \\
3
\end{pmatrix},
\begin{pmatrix}
1 \\
2 \\
5
\end{pmatrix},
\begin{pmatrix}
3 \\
1 \\
5
\end{pmatrix}
\) in \(\mathbb{R}^3\). Are these vectors linearly independent? Do they span \(\mathbb{R}^3\)? Do they form a basis for \(\mathbb{R}^3\)?

**Answer:** Row-reducing the matrix 
\[
A = \begin{pmatrix}
1 & 1 & 3 \\
1 & 2 & 1 \\
3 & 5 & 5
\end{pmatrix},
\]
whose columns are the vectors in question, yields 
\[
\begin{pmatrix}
1 & 0 & 5 \\
0 & 1 & -2 \\
0 & 0 & 0
\end{pmatrix}
\]
so the vectors are not linearly independent (since there are more than two columns), they do not span (since there are more than two rows), and they do not form a basis.

2. (15 points) Let \(V = \mathbb{R}_2[t]\) be the space of quadratic polynomials in a variable \(t\) and consider the linear transformation \(L(p) = (t+1)p'(t)\) from \(V\) to itself, where \(p'(t)\) is the derivative of \(p(t)\). Find the matrix of this linear transformation with respect to the (standard) basis \(\{1, t, t^2\}\).

**Answer:** Since \(L(1) = 0\), \(L(t) = t + 1\) and \(L(t^2) = 2t(t + 1) = 2t^2 + 2t\), our matrix is 
\[
\begin{pmatrix}
[ L(1)]_{\mathcal{E}} & [ L(t)]_{\mathcal{E}} & [ L(t^2)]_{\mathcal{E}} 
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 2 \\
0 & 1 & 2 \\
0 & 0 & 2
\end{pmatrix}.
\]

3. Let \(A = \begin{pmatrix}
1 & -1 & -1 & 1 & 8 \\
1 & 2 & 8 & 3 & 7 \\
1 & 2 & 8 & -2 & -28 \\
1 & 5 & 17 & 0 & -29
\end{pmatrix}\). \(A\) is row-equivalent to 
\[
\begin{pmatrix}
1 & 0 & 2 & 0 & -4 \\
0 & 1 & 3 & 0 & -5 \\
0 & 0 & 0 & 1 & 7 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
a) Find a basis for the null space of $A$.

**Answer:** The equations $A_{rref}x = 0$ are essentially:

\[
\begin{align*}
    x_1 &= -2x_3 + 4x_5 \\
    x_2 &= -3x_3 + 5x_5 \\
    x_3 &= x_3 \\
    x_4 &= -7x_5 \\
    x_5 &= x_5,
\end{align*}
\]

so our basis is

\[
\left\{\begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 0 \\ -7 \\ 1 \end{pmatrix}\right\}.
\]

b) Find a basis for the column space of $A$.

**Answer:** Since there are pivots in the first, second and fourth columns, we want the first, second and fourth columns of $A$, namely

\[
\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \end{pmatrix}\right\}
\]

4. a) In $\mathbb{R}^2$, let $\mathcal{B} = \left\{\begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \end{pmatrix}\right\}$ be a basis, and let $x = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Let $\mathcal{E}$ be the standard basis. Compute the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$ and compute the coordinates of $x$ in the $\mathcal{B}$ basis.

**Answer:** $P_{\mathcal{E}\mathcal{B}} = (b_1, b_2) = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$. $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} -8 \\ 5 \\ -3 \end{pmatrix}$. (Remember the formula for the inverse of a $2 \times 2$ matrix!) Then $[x]_\mathcal{B} = P_{\mathcal{B}\mathcal{E}}[x]_\mathcal{E} = \begin{pmatrix} -8 \\ 5 \\ -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -18 \\ 11 \end{pmatrix}$. We can check that we did things right by computing $-18b_1 + 11b_2$ and seeing that it really is $x$.

b) In $\mathbb{R}_1[t]$, let $\mathcal{D} = \{3 + 5t, 5 + 8t\}$ and let $p(t) = 1 + t$. Compute $[p]_\mathcal{D}$.

**Answer:** This is similar, as $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$ are the same as in part (a). Since $[p]_\mathcal{E} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $[p]_\mathcal{B} = P_{\mathcal{B}\mathcal{E}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. As a check, $-3(3 + 5t) + 2(5 + 8t)$ does equal $1 + t$. 

2
5. a) Find the characteristic polynomial of \( \begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix} \). (You do not have to compute the eigenvalues or eigenvectors).

**Answer:**
\[
p_A(\lambda) = \det \begin{pmatrix} \lambda - 3 & -2 \\ -5 & \lambda - 1 \end{pmatrix} = (\lambda - 3)(\lambda - 1) - 10 = \lambda^2 - 4\lambda - 7.
\]

b) Find the eigenvalues of \( \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \). (You do not have to find the eigenvectors).

**Answer:** In this case, \( p_A(\lambda) = (\lambda - 1)^2 + 4 = \lambda^2 - 2\lambda + 5 \). From the quadratic formula, or by completing the square (which was complete to begin with), the roots are \( 1 \pm 2i \).

c) \( \lambda = 2 \) is one of the eigenvalues of \( \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \). Find a basis for the corresponding eigenspace.

**Answer:**
\[
2I - A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}
\]
row-reduces to \( \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), hence the equations \( x_1 = -x_2 - x_3, x_2 = x_2, x_3 = x_3 \), and basis \( \{ (-1, 1, 0)^T, (-1, 0, 1)^T \} \) for \( E_2 \).