M346 Second Midterm Exam, April 7, 2011

1) The matrix $A = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -4$, with eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Suppose that $\mathbf{x}(n)$ satisfies the system of equations $\mathbf{x}(n+1) = A\mathbf{x}(n)$ for all $n \geq 0$.

a) If $\mathbf{x}(0)$ is "random" (meaning any nonzero vector that isn't an eigenvector of A), compute the limits $\lim_{n\to\infty} \frac{x_1(n)}{x_2(n)}$ and $\lim_{n\to\infty} \frac{x_1(n+1)}{x_1(n)}$. In other words, what is the asymptotic direction of $\mathbf{x}(n)$ and the asymptotic growth rate?

b) Now suppose that $\mathbf{x}(0) = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$. Find $\mathbf{x}(n)$ exactly for all n.

2) Let $A = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$, exactly as in problem 1. Suppose that $\mathbf{x}(t)$ satisfies the differential equation $d\mathbf{x}/dt = A\mathbf{x}$.

a) How many stable and how many unstable modes does this system have? What is the dominant eigenvalue, and what is the dominant eigenvector? For typical initial conditions, compute $\lim_{t\to\infty} x_1(t)/x_2(t)$.

b) Solve the differential equations with initial conditions $\mathbf{x}(0) = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

3) Let $A = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$, exactly as in problems 1 and 2, only now consider the second order differential equations $d^2\mathbf{x}/dt^2 = A\mathbf{x}$. Write down the most general solution to these equations. (Leave your answer in terms of arbitrary constants, not in terms of initial conditions. I'm not giving you the initial conditions.)

b) Find a solution corresponding to the initial conditions $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $d\mathbf{x}/dt(0) = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

4. Consider the nonlinear system of differential equations

$$\frac{dx_1}{dt} = x_1(3 - x_1 - 2x_2); \qquad \frac{dx_2}{dt} = x_2(3 - 2x_1 - x_2)$$

These equations describe the fierce competition between two species for similar resources, where $x_1(t)$ and $x_2(t)$ are the populations of the two species at time t. The fixed points are at (0,0), (3,0), (0,3) and (1,1).

a) (16 pts) For each fixed point, determine how many stable and unstable modes there are.

b) (4 pts) Describe the possible long-term behavior of the system.

5. a) Find an orthogonal basis for the column space of $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & -2 \\ 4 & 5 & 7 \end{pmatrix}$, where we are using the standard inner product for $\mathbb{R}^4$ (and subspaces of $\mathbb{R}^4$).

b) Find an orthogonal basis for the column space of $\begin{pmatrix} 1 & 0 \\ i & 1-i \\ i & -1-3i \\ 2 & -5 \end{pmatrix}$, using the standard inner product for $\mathbb{C}^4$.