M346 Final Exam, December 15, 2005

1. In $\mathbb{R}_2[t]$, consider the standard basis $\mathcal{E} = 1, t, t^2$ and the alternate basis $\mathcal{B} = 1 - t + 3t^2, 2t - t^2, -t + t^2$.

a) Are the vectors $\mathbf{b}_1 = 1 + 3t + 2t^2$, $\mathbf{b}_2 = 2 + t + t^2$, $\mathbf{b}_3 = 7 + 4t + 5t^2$ linearly independent? Do they span $\mathbb{R}_2[t]$?

The coordinates of these vectors in the standard basis are $(1, 3, 2)^T$, $(2, 1, 1)^T$ and $(7, 4, 5)^T$. Since the matrix $\begin{pmatrix} 1 & 2 & 7 \\ 3 & 1 & 4 \\ 2 & 1 & 5 \end{pmatrix}$ is invertible (row-reduce it, or take its determinant, which is -6), those three coordinates form a basis for $\mathbb{R}^3$, so the original vectors form a basis for $\mathbb{R}_2[t]$.

b) Find the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$.

The matrix $P_{\mathcal{E}\mathcal{B}}$ is made from the coordinates of the $\mathcal{B}$ vectors in the $\mathcal{E}$ basis, and is $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix}$. $P_{\mathcal{B}\mathcal{E}}$ is the inverse of this matrix, namely $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ -5 & 1 & 2 \end{pmatrix}$.

c) Find $[\mathbf{x}]_{\mathcal{B}}$, where $\mathbf{x} = 1 + 10t + 100t^2$.

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\mathbf{x}]_{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ -5 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 10 \\ 100 \end{pmatrix} = \begin{pmatrix} 1 \\ 108 \\ 205 \end{pmatrix}$$

2. On $M_{2,2}$, consider the linear transformation $L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

a) Find the matrix of L with respect to the standard basis
$\mathcal{E} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

Since $L(\mathbf{e}_1) = \mathbf{e}_4, L(\mathbf{e}_2) = -\mathbf{e}_2, L(\mathbf{e}_3) = -\mathbf{e}_3$ and $L(\mathbf{e}_4) = \mathbf{e}_1$ we have
$[L]_{\mathcal{E}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

b) Find a basis for the eigenspace E_{-1}.

We are looking for the null space of $L-(-1)I = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. This is one equation in 4 unknowns, namely $x_1 = -x_4$. The three free variables are $x_{2,3,4}$, and our basis vectors are $(0,1,0,0)^T$, $(0,0,1,0)^T$, and $(-1,0,0,1)^T$. These are eigenvectors of the matrix $[L]_{\mathcal{E}}$ and correspond to the eigenvectors $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ of the operator L.

3. a) Find the eigenvalues of the matrix

$$\begin{pmatrix} 3 & 2 & 3 & 1 & 4 & 1 \\ -2 & 3 & 2 & 1 & 7 & 1 \\ 0 & 0 & 4 & 1 & 4 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

You do not need to compute the eigenvectors.

This is block-triangular. The upper left 2×2 block has eigenvalues $3 \pm 2i$, the middle block has eigenvalue 4, and the lower right 3×3 block has eigenvalues $2, 2 \pm \sqrt{2}$. (The upper block is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, which always has eigenvalues $a \pm bi$, and the lower block should be familiar from homework. It also isn't too hard to compute directly.)

b) Compute A^{10}, where $A = \begin{pmatrix} 2/5 & -6/5 \\ -6/5 & -7/5 \end{pmatrix}$.

The eigenvalues of A are 1 and -2, with eigenvectors $(2,-1)^T$ and $(1,2)^T$. Thus $A = PDP^{-1}$, where $P = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ and $P^{-1} = (1/5)\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$. We then compute

$$A^{10} = PD^{10}P^{-1} = (1/5)\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1024 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = (1/5)\begin{pmatrix} 1028 & 2046 \\ 2046 & 4097 \end{pmatrix}.$$

4. Overpopulation in fairyland. Fairies come in two varieties: immortal and mortal. Every year, each immortal fairy gives birth to three mortal fairies. Every year, each mortal fairy gives birth to two immortal fairies, and then dies. (Immortal fairies never die, hence the name.)

a) Write down a set of equations to describe the evolution of the immortal and mortal populations.

Let $I(n)$ and $M(n)$ be the populations of immortal and mortal fairies in year n, and let $\mathbf{x} = (I, M)^T$. Then $\mathbf{x}(n+1) = A\mathbf{x}(n)$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$, since each year every immortal fairy turns into one immortal (itself) and three mortals, while each mortal turns into two immortals.

b) How fast does the overall fairy population grow? After a long time, what will be the (limiting) ratio of immortal to mortal fairies?

The eigenvalues of A are 3 and -2, with eigenvectors $(1,1)^T$ and $2,-3)^T$. Thus the population grows asymptotically as 3^n, and the ratio I:M approaches 1:1.

c) If in year zero there are 11 immortal fairies and 1 mortal fairy, how many fairies of each type will there be in year n?

$\mathbf{x}(0) = (11,1)^T = 7(1,1)^T + 2(2,-3)^T$, so $\mathbf{x}(n) = 7(3^n)(1,1)^T + 2(-2)^n(2,-3)^T$. In other words, $I(n) = 7(3)^n + 4(-2)^n$ and $M(n) = 7(3)^n - 6(-2)^n$.

5. Let $\mathbf{x}(t)$ be a *complex* 2-vector that satisfies the differential equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \mathbf{x}.$$

a) Find the general solution to this system of equations. How many stable/neutral/unstable modes are there?

The eigenvalues of the matrix are 2 (unstable) and 0 (neutral), with eigenvectors $(1,-i)^T$ and $(1,i)^T$, respectively. The general solution is $\mathbf{x}(t) = c_2 e^{2t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_0 \begin{pmatrix} 1 \\ i \end{pmatrix}$. There are no stable modes, but there is one unstable and one neutral.

b) If $\mathbf{x}(0) = (1,0)^T$, find $\mathbf{x}(t)$.

$\mathbf{x}(0) = [(1,i)T + (1,-i)^T]/2$, so $\mathbf{x}(t) = [e^{2t} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \begin{pmatrix} 1 \\ i \end{pmatrix}]/2$.

6. Gram-Schmidt. Convert the following collections of vectors to orthogonal collections using the Gram-Schmidt process.

a) In $\mathbb{C}^3$ with the usual inner product, $\mathbf{x}_1 = (1, 1+i, i)^T$, $\mathbf{x}_2 = (3, 2+2i, i)^T$, $\mathbf{x}_3 = (12, -4-4i, 0)^T$.

$\mathbf{y}_1 = \mathbf{x}_1 = (1, 1+i, i)^T$, and we compute $\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle = 4$, $\langle \mathbf{y}_1 \mathbf{x}_2 \rangle = 8$, $\langle \mathbf{y}_1 | \mathbf{x}_3 \rangle = 4$. $\mathbf{y}_2 = \mathbf{x}_2 - (\langle \mathbf{y}_1 | \mathbf{x}_2 \rangle / \langle \mathbf{y}_1 | \mathbf{y}_1 \rangle)\mathbf{y}_1 = \mathbf{x}_2 - 2\mathbf{y}_1 = (1, 0, -i)^T$. We then compute $\langle \mathbf{y}_2|\mathbf{y}_2\rangle = 2$ and $\langle \mathbf{y}_2|\mathbf{x}_3\rangle = 12$, so $\mathbf{y}_3 = \mathbf{x}_3 - (\langle \mathbf{y}_1|\mathbf{x}_2\rangle/\langle \mathbf{y}_1|\mathbf{y}_1\rangle)\mathbf{y}_1 - (\langle \mathbf{y}_2|\mathbf{x}_2\rangle/\langle \mathbf{y}_2|\mathbf{y}_2\rangle)\mathbf{y}_2 = \mathbf{x}_3 - \mathbf{y}_1 - 6\mathbf{y}_2 = 5(2, -1-i, i)^T$.

b) In $\mathbb{R}_2[t]$, with the inner product $\langle f|g\rangle = \int_0^2 f(t)g(t)dt$, $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = t$, $\mathbf{x}_3 = t^2$.

$\mathbf{y}_1 = \mathbf{x}_1 = 1$, and we compute $\langle \mathbf{y}_1|\mathbf{y}_1\rangle = 2$, $\langle \mathbf{y}_1 \mathbf{x}_2\rangle = 2$, $\langle \mathbf{y}_1|\mathbf{x}_3\rangle = 8/3$. $\mathbf{y}_2 = \mathbf{x}_2 - (\langle \mathbf{y}_1|\mathbf{x}_2\rangle/\langle \mathbf{y}_1|\mathbf{y}_1\rangle)\mathbf{y}_1 = \mathbf{x}_2 - \mathbf{y}_1 = t-1$. We then compute $\langle \mathbf{y}_2|\mathbf{y}_2\rangle = 2/3$ and $\langle \mathbf{y}_2|\mathbf{x}_3\rangle = /3$, so $\mathbf{y}_3 = \mathbf{x}_3 - (\langle \mathbf{y}_1|\mathbf{x}_2\rangle/\langle \mathbf{y}_1|\mathbf{y}_1\rangle)\mathbf{y}_1 - (\langle \mathbf{y}_2|\mathbf{x}_2\rangle/\langle \mathbf{y}_2|\mathbf{y}_2\rangle)\mathbf{y}_2 = \mathbf{x}_3 - (4/3)\mathbf{y}_1 - 2\mathbf{y}_2 = t^2 - 2t + 2/3$.

7. Rotations. Let $A = \frac{\pi}{3}\begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix}$ and let $R = \exp(A)$. Since A is anti-symmetric, R is orthogonal, and is a rotation in $\mathbb{R}^3$.

a) Find the axis for the rotation R, and the angle of rotation.

The eigenvalues of A are $0, i\pi, -i\pi$, and the eigenvector with eigenvalue 0 is $(2,-1,2)^T$. This means that the eigenvalues of R are $1, e^{i\pi}, e^{-i\pi}$, and that the eigenvector with eigenvalue 1 is $(2,-1,2)$. Thus we have a rotation by π about the $(2,-1,2)^T$ axis.

b) Compute R.

Note that $e^{i\pi} = e^{-i\pi} = -1$, so $R = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}P^{-1}$. We could compute the matrix P and its inverse (and most of you probably did it that way), but there's a cute shortcut, using the fact that $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - I$, so $R = 2P_{\mathbf{b}_1} - I = \frac{2}{9}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}(2,-1,2) - I = \begin{pmatrix} -1/9 & -4/9 & 8/9 \\ -4/9 & -7/9 & -4/9 \\ -1/9 & -4/9 & -1/9 \end{pmatrix}$.

8. a) Compute the solution to the wave equation: $\partial^2 f/\partial t^2 = \partial^2 f/\partial x^2$ on the whole line with with initial conditions $f(x,0) = \frac{1}{1+x^2}$, $\frac{\partial f}{\partial t}(x,0) = \frac{3x}{(1+x^2)^2}$.

Note that $v = 1$ and that $\int g_0(x)dx = -3/(1+x^2)$. Since $h_1 = (f_0 - \int g_0)/2$ and $h_2 = (f_0 + \int g_0)/2$, we have $h_1(x) = 2/(1+x^2)$ and $h_2(x) = -1/(1+x^2)$. Finally,

$$f(x,t) = h_1(x-t) + h_2(x+t) = \frac{2}{1+(x-t)^2} - \frac{1}{1+(x+t)^2}$$

b) Compute the solution to the wave equation on the half-line $[0, \infty)$ with Dirichlet boundary conditions at $x = 0$ and with initial conditions $f(x, 0) = x/(1 + x^2)$, $\dot{f}(x, 0) = 0$.

Since $g_0 = 0$, we have $h_1 = h_2 = f_0/2 = x/(2 + 2x^2)$. This applies only to $x > 0$, and $h_1(x) = h_2(x) = 0$ for $x < 0$. We then take $f(x, t) = h_1(x - t) + h_2(x + t) - h_1(-x - t) - h_2(-x + t)$. Conveniently, $h_1(x - t) - h_2(-x + t) = \frac{1}{2}\frac{x-t}{1+(x-t)^2}$ for all x, and likewise $h_2(x + t) - h_1(-x - t) = \frac{1}{2}\frac{x+t}{1+(x+t)^2}$, so

$$f(x, t) = \frac{1}{2}\left(\frac{x - t}{1 + (x - t)^2} + \frac{x + t}{1 + (x + t)^2}\right)$$

Another way of saying this is that f_0 is already odd, so the expression in the displayed equation solves the wave equation for all values of x and meets the boundary condition.

9. Consider a vibrating string with $v = L = 1$, with $f(x, 0) = f_0(x)$ and $\dot{f}(x, 0) = g_0(x)$, where $f_0(x) = \sum n^{-3} \sin(n\pi x)$ and $g_0(x) = \sum 2\pi n^{-2} \sin(n\pi x)$.

a) Find $f(x, t)$. You can write your solution as an infinite sum, but the coefficients should be explicit.

The general solution is $f(x, t) = \sum_n (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \sin(n\pi x)$. At $t = 0$ we have $f_0(x) = \sum a_n \sin(n\pi x)$, so $a_n = n^{-3}$, and $g_0(x) = \sum n\pi b_n \sin(n\pi x)$, so $b_n = 2n^{-3}$. Thus our solution is

$$f(x, t) = \sum_{n=1}^{\infty} n^{-3}(\cos(n\pi t) + 2\sin(n\pi t)) \sin(n\pi x)$$

b) At what time(s) will $f(x, t) = f_0(x)$ (for all x)?

We need to have $\cos(n\pi t) + 2\sin(n\pi t) = 1$ for all values of n. There are two values of θ for which $\cos(\theta) + 2\sin(\theta) = 1$, namely $\theta = 0$ and $\theta = 2\tan^{-1}(2)$. However, the second solution gives different values of t for each value of n, so the only way to get it to work for all values of n is to have $n\pi t$ be a multiple of 2π, i.e., for t to be a multiple of 2. [Anybody who gets this solution gets full credit; if you correctly analyzed the $2\tan^{-1}(2)$ possibility you get extra credit]

10. Consider the periodic function, with period 1, that equals the following on the interval $[0, 1]$:

$$f(x) = \begin{cases} 1 & \text{if } 1/4 < x < 3/4 \\ 0 & \text{otherwise} \end{cases}$$

a) At what rate do the Fourier coefficients $\hat{f}_n$ decay with n? (This should require almost no calculation).

Since $f(x)$ is discontinuous, the coefficients decay as n^{-1}.

b) Compute $\hat{f}_n$ for all n. [Does your answer agree with (a)?]

For $n = 0$, we have $\hat{f}_0 = \int_0^1 f(x)dx = 1/2$. For all other values of n we have $\hat{f}_n = \int_0^1 f(x)\exp(-2\pi nx)dx = \int_{1/4}^{3/4} \exp(-2\pi inx)dx = \frac{ie^{-2\pi inx}}{2\pi n}\Big|_{1/4}^{3/4}$. Since $\exp(-2\pi inx)$ is periodic with period 1, this equals $\frac{ie^{-2\pi inx}}{2\pi n}\Big|_{1/4}^{-1/4} = \frac{1}{2\pi n}(e^{in\pi/2} - e^{-in\pi/2}) = -\sin(n\pi/2)/n\pi$. This equals 0 if n is even, and equals $(-1)^{k+1}/n\pi$ if $n = 2k+1$.

c) Compute $\sum_{n=-\infty}^{\infty} |\hat{f}_n|^2$.

The easy way to do this (and the intended way!) is to note that $\sum |\hat{f}_n|^2 = \langle f|f\rangle = \int_0^1 f(x)^2 dx = 1/2$. A similar problem appeared in a practice final. The hard way is to do the infinite sum directly from (b) using the identity $\sum_{n>0 \text{ odd}} n^{-2} = \pi^2/8$, which you derived in the homework.