

M346 Final Exam, December 15, 2005

1. In $\mathbb{R}_2[t]$, consider the standard basis $\mathcal{E} = 1, t, t^2$ and the alternate basis $\mathcal{B} = 1 - t + 3t^2, 2t - t^2, -t + t^2$.

a) Are the vectors $\mathbf{b}_1 = 1 + 3t + 2t^2$, $\mathbf{b}_2 = 2 + t + t^2$, $\mathbf{b}_3 = 7 + 4t + 5t^2$ linearly independent? Do they span $\mathbb{R}_2[t]$?

b) Find the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$.

c) Find $[\mathbf{x}]_{\mathcal{B}}$, where $\mathbf{x} = 1 + 10t + 100t^2$.

2. On $M_{2,2}$, consider the linear transformation $L \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

a) Find the matrix of L with respect to the standard basis

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

b) Find a basis for the eigenspace E_{-1} .

3. a) Find the eigenvalues of the matrix

$$\begin{pmatrix} 3 & 2 & 3 & 1 & 4 & 1 \\ -2 & 3 & 2 & 1 & 7 & 1 \\ 0 & 0 & 4 & 1 & 4 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

You do not need to compute the eigenvectors.

b) Compute A^{10} , where $A = \begin{pmatrix} 2/5 & -6/5 \\ -6/5 & -7/5 \end{pmatrix}$.

4. Overpopulation in fairyland. Fairies come in two varieties: immortal and mortal. Every year, each immortal fairy gives birth to three mortal fairies. Every year, each mortal fairy gives birth to two immortal fairies, and then dies. (Immortal fairies never die, hence the name.)

a) Write down a set of equations to describe the evolution of the immortal and mortal populations.

b) How fast does the overall fairy population grow? After a long time, what will be the (limiting) ratio of immortal to mortal fairies?

c) If in year zero there are 11 immortal fairies and 1 mortal fairy, how many fairies of each type will there be in year n ?

5. Let $\mathbf{x}(t)$ be a *complex* 2-vector that satisfies the differential equation

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \mathbf{x}.$$

a) Find the general solution to this system of equations. How many stable/neutral/unstable modes are there?

b) If $\mathbf{x}(0) = (1, 0)^T$, find $\mathbf{x}(t)$.

6. Gram-Schmidt. Convert the following collections of vectors to orthogonal collections using the Gram-Schmidt process.

a) In \mathbb{C}^3 with the usual inner product, $\mathbf{x}_1 = (1, 1+i, i)^T$, $\mathbf{x}_2 = (3, 2+2i, i)^T$, $\mathbf{x}_3 = (12, -4-4i, 0)^T$.

b) In $\mathbb{R}_2[t]$, with the inner product $\langle f|g \rangle = \int_0^2 f(t)g(t)dt$, $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = t$, $\mathbf{x}_3 = t^2$.

7. Rotations. Let $A = \frac{\pi}{3} \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix}$ and let $R = \exp(A)$. Since A is anti-symmetric, R is orthogonal, and is a rotation in \mathbb{R}^3 .

a) Find the axis for the rotation R , and the angle of rotation.

b) Compute R .

8. a) Compute the solution to the wave equation: $\partial^2 f / \partial t^2 = \partial^2 f / \partial x^2$ on the whole line with initial conditions $f(x, 0) = \frac{1}{1+x^2}$, $\frac{\partial f}{\partial t}(x, 0) = \frac{3x}{(1+x^2)^2}$.

b) Compute the solution to the wave equation on the half-line $[0, \infty)$ with Dirichlet boundary conditions at $x = 0$ and with initial conditions $f(x, 0) = x/(1+x^2)$, $\dot{f}(x, 0) = 0$.

9. Consider a vibrating string with $v = L = 1$, with $f(x, 0) = f_0(x)$ and $\dot{f}(x, 0) = g_0(x)$, where $f_0(x) = \sum n^{-3} \sin(n\pi x)$ and $g_0(x) = \sum 2\pi n^{-2} \sin(n\pi x)$.

a) Find $f(x, t)$. You can write your solution as an infinite sum, but the coefficients should be explicit.

b) At what time(s) will $f(x, t) = f_0(x)$ (for all x)?

10. Consider the periodic function, with period 1, that equals the following on the interval $[0, 1]$:

$$f(x) = \begin{cases} 1 & \text{if } 1/4 < x < 3/4 \\ 0 & \text{otherwise} \end{cases}$$

a) At what rate do the Fourier coefficients \hat{f}_n decay with n ? (This should require almost no calculation).

b) Compute \hat{f}_n for all n . [Does your answer agree with (a)?]

c) Compute $\sum_{n=-\infty}^{\infty} |\hat{f}_n|^2$.