A is an $n \times n$ matrix.

To solve $Ax = b$, row-reduce

$$(A \mid b) \longrightarrow (U \mid c)$$

where U is in REF.

Typical case: $U = \begin{pmatrix} d_1 & & & & \\ 0 & d_2 & & & \\ 0 & 0 & \ddots & & \\ 0 & 0 & 0 & 0 & d_n \end{pmatrix} =$ upper triangular.

Each row gives the corresponding variable in terms of the subsequent ones.

$$d_1 x_1 + (\text{stuff}) = c_1$$
$$d_2 x_2 + (\text{stuff}) = c_2$$
$$\vdots$$
$$d_n x_n = c_n$$

Back Substitution.

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b-2a \end{pmatrix}$$

$$E_{21} \quad \underline{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}} \begin{pmatrix} a & c \\ \underline{b} & d \end{pmatrix} = \begin{pmatrix} a & c \\ b-2a & d-2c \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \begin{pmatrix} a & c & e \\ b-2a & d-2c & f-2e \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \text{---} \; r_1 \; \text{---} \\ \text{---} \; r_2 \; \text{---} \end{pmatrix} = \begin{pmatrix} \text{---} \; r_1 \; \text{---} \\ \text{---} \; r_2 - 2r_1 \; \text{---} \end{pmatrix}$$

$$E_{ij} = \begin{array}{cc} & \begin{array}{ccccc} & j & & & \end{array} \\ i & \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & -l_{ij} & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array}$$

$$E_{ij} \begin{pmatrix} \text{---} \; r_1 \; \text{---} \\ \text{---} \; r_2 \; \text{---} \\ \vdots \\ \text{---} \; r_n \; \text{---} \end{pmatrix} = \begin{pmatrix} \text{Same} \\ \text{except} \\ r_i \to r_i - l_{ij} r_j \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} b & d \\ a & c \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} - & r_1 & - \\ - & r_2 & - \end{pmatrix} = \begin{pmatrix} - & r_2 & - \\ - & r_1 & - \end{pmatrix}$$

$$P_{ij} = \begin{array}{c} \begin{array}{ccccccccc} & & & i & & & j & & \\ & & & \downarrow & & & \downarrow & & \end{array} \\ \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & 1 & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & 1 & & & 0 & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \end{pmatrix} \end{array}$$

If E is a matrix that does a row operation,

$E = EI =$ that row operation applied to I.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ rescales 2nd row by 7.

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ **IN THIS CASE** order doesn't matter.

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

BUT

$$\underset{A}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}} \underset{B}{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 12 & 4 & 1 \end{pmatrix}$$

$$\begin{array}{l c} & \begin{pmatrix} 3 & 15 & 24 \\ 7 & 97 & -3 \\ 14 & \pi & \sqrt{2} \end{pmatrix} \\ \text{first} \rightarrow & \\ \text{2nd} \rightarrow & \end{array}$$

3rd.

l_{21}

l_{31}

l_{32}

U

What is a $m \times n$ matrix?

1) An array of numbers.

2) A list of columns (n vectors in $\mathbb{R}^m$)

3) A list of rows (m equations in $\mathbb{R}^n$)

4) An operation on vectors.

$$\begin{aligned} A:\ \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\longrightarrow Ax \end{aligned}$$

5) An operation on ~~a~~ $n \times p$ matrices.

$${}_m\left(\overset{n}{A}\right)\,{}_n\left(\overset{p}{B}\right) = {}_m\left(\overset{p}{AB}\right)$$

What is AB?

$$\left[A = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} \quad B = \begin{pmatrix} b_1 & \cdots & b_p \end{pmatrix} \right]$$

$AB =$ do B, then do A

$= A$ applied to B

$$= \begin{pmatrix} A(b_1) & \cdots & A(b_p) \end{pmatrix}$$

$$= \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} \begin{pmatrix} b_1 & \cdots & b_p \end{pmatrix}$$

$=$ Matrix whose ij entry is $r_i b_j$

$$= \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$= \begin{pmatrix} r_1 B \\ r_2 B \\ \vdots \\ r_m B \end{pmatrix}$$

What is $A\vec{x}$?

1) Linear combination of columns of A with coefficients given by x.

$$= \sum_j x_j \vec{a_j}$$

2) Vector in $\mathbb{R}^m$ whose i-th entry is

$$\sum_j a_{ij} x_j = r_i x$$

= function of x given by i-th row.

3) Output of A operator.

Partitioned matrices

$$M = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \quad A \qquad\qquad\qquad B \\ \left(\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \end{array}\right) \\ \quad C \qquad\qquad\qquad D \end{array} = \begin{array}{c|cc} & 3 & 2 \\ \hline 2 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$}} \\ 1 & & \end{array}$$

$$\vec{v} = \left(\begin{array}{c} 3 \\ 1 \\ 4 \\ \hline 2 \\ 8 \end{array}\right) = \begin{array}{c|c} & 1 \\ \hline 3 & \multirow{2}{*}{$\left(\begin{array}{c} \vec{x} \\ \hline \vec{y} \end{array}\right)$} \\ 2 & \end{array}$$

$$M\vec{v} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{array}{c|c} & 1 \\ \hline 2 & \multirow{2}{*}{$\left(\begin{array}{c} Ax + By \\ \hline Cx + Dy \end{array}\right)$} \\ 1 & \end{array}$$

$$\begin{array}{c|c} & 2n \\ \hline 2n & \left(\begin{array}{c|c} I & B \\ \hline O & I \end{array}\right) \end{array} \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$= I_{2n}$$