An $m \times n$ matrix is a map

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

A square $n \times n$ matrix is a map

$$\mathbb{R}^n \circlearrowleft$$

$$A^k = A \text{ appied } k \text{ times.}$$

$$\neq \begin{pmatrix} a_{11}^k & \cdots & a_{1n}^k \\ a_{n1}^k & \cdots & a_{nn}^k \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} - r_1 - \\ - r_2 - \end{pmatrix} = \begin{pmatrix} r_1 \\ r_1 + r_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^k = \text{add 1st entry to the 2nd } k \text{ times}$$

$$= \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

$$\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\2&1\end{pmatrix}$$

$$\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\2&1\end{pmatrix}=\begin{pmatrix}1&0\\3&1\end{pmatrix}$$

$A=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ = Swap

$$A^k=\begin{cases}\begin{pmatrix}1&0\\0&1\end{pmatrix} & k \text{ even}\\ \begin{pmatrix}0&1\\1&0\end{pmatrix} & k \text{ odd}\end{cases}$$

$A=\begin{pmatrix}2&0\\0&1\end{pmatrix}$ = doubles first entry/row

A^k = multiply first entry/row by 2^k

$$=\begin{pmatrix}2^k&0\\0&1\end{pmatrix}$$

Fact: If A is a diagonal matrix

$A=\begin{pmatrix}d_1&&0\\&\ddots&\\0&&d_n\end{pmatrix}$, then $A^k=\begin{pmatrix}d_1^k&&0\\&\ddots&\\0&&d_n^k\end{pmatrix}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow[\text{to } r_2]{\text{add } r_1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{swap}} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \end{pmatrix} = \begin{pmatrix} - & r_1 + r_2 & - \\ - & r_1 & - \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}$$

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+1} = F_k + F_{k-1}$$

$$A^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}$$

$$F_{k+2} = F_{k+1} + F_k$$

A^{-1} = inverse of A

= undoes A.

If $Ax = y$, $A^{-1}y = x$

$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ does not have an inverse.

$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad$ $A^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = ?$ (no good answer)

$A\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Thm ~~If~~ Let A ~~is~~ be a square matrix ($n \times n$).

The following are equivalent:

a) A has an inverse.

b) $A\vec{x} = \vec{b}$ has exactly one soln for each $\vec{b}$

c) A row-reduces to a triangular matrix

d) Row-reducing yields n pivots

e) No rows of 0's at bottom of row-~~red~~echelon form.

f) No free variables

g) Only solution to $Ax = 0$ is $x = 0$

h) The only way to write

$$x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n = 0 \text{ is with } x_1 = x_2 = \cdots x_n = 0,$$

where $A = (\vec{a}_1 \cdots \vec{a}_n)$

i) Columns of A are <u>linearly independent</u>

j) No column is a linear combination of the others.

k) Every $\vec{b} \in \mathbb{R}^n$ is a linear comb of the columns

l) Span (columns) $= \mathbb{R}^n$.

Let $\mathcal{C} = \{\vec{v}_1 \dots \vec{v}_k\}$ be a bunch of vectors in $\mathbb{R}^n$.

<u>Def</u> The collection is <u>linearly independent</u> if the only way to get $\vec{0}$ as a linear combination is as $0\vec{v}_1 + \dots + 0\vec{v}_k$.

<u>Def</u> $\text{Span}(\mathcal{C}) = \{$all linear combinations of $\vec{v}_1, \dots \vec{v}_k\}$.

<u>Thm</u> Linear. Independence $\Leftrightarrow$ no redundancy

$\Leftrightarrow$ no vector is lin comb of others

$\Leftrightarrow$ no vector is lin comb of previous ones.

Examples of inverses

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -l_{ij} & & & \ddots & \\ & & & & 1 \end{pmatrix} = \text{subtract } l_{ij} r_j \text{ from } r_i$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \end{pmatrix} = \begin{pmatrix} - r_1 - \\ - r_2 - \\ - r_3 - 3r_2 - \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ -l_{ij} & & \ddots & \\ & & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ l_{ij} & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{d_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{d_n} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

To undo a swap, do it again!

<u>Thm</u> If A and B have inverses, then so does AB, and $(AB)^{-1} = B^{-1}A^{-1}$

<u>Pf 1</u> To undo "B, then A", first undo A, then undo B.

<u>Pf 2</u>

$$B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$$

$$AB\,B^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

How to compute A^{-1}:

1) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 2×2,

$$A^{-1} = \begin{cases} \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} & \text{if } ad-bc \neq 0 \\ \text{DNE} & \text{if } ad-bc = 0 \end{cases}$$

$$A^{-1}A = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$$

If A is $n \times n$ matrix

1) Start with $(A \mid I)$
2) Apply row operations to get to

$$(U \mid M_0),$$ where U is upper-triangular.

3) Apply additional row operations to turn U into I.

$$(A \mid I) \to (U \mid M_0) \to (I \mid M)$$

4) $A^{-1} = M$.