How to find A^{-1}

Case I: A is 2×2.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

if $ad-bc \neq 0$

Cas II: A is bigger than 2×2.

1) Start with $(A \mid I)$
2) Apply row operations to get $(U \mid M_0)$
3) Apply more " " to get $(I \mid M)$
4) $A^{-1} = M$

$$(A \mid I) \longrightarrow (I \mid M)$$

$$E_N \cdots E_2 E_1 (A \mid I) = (I \mid M)$$

$$\left.\begin{aligned} E_N \cdots E_1 A &= I \\ E_N \cdots E_1 I &= M \end{aligned}\right\} \rightarrow \quad MA = I$$

$$(A \quad I) = E_1^{-1} E_2^{-1} \cdots E_N^{-1} (I \quad M)$$

$$A = E_1^{-1} \cdots E_N^{-1}$$

$$I = E_1^{-1} \cdots E_N^{-1} M = AM$$

Ex $\quad A = \begin{pmatrix} 2 & 6 & -3 \\ 2 & 9 & 6 \\ 4 & 18 & 11 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & 6 & -3 & 1 & 0 & 0 \\ 2 & 9 & 6 & 0 & 1 & 0 \\ 4 & 18 & 11 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow \quad \begin{array}{l} r_2 \to r_2 - r_1 \\ r_3 \to r_3 - 2r_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 6 & -3 & 1 & 0 & 0 \\ 0 & 3 & 9 & -1 & 1 & 0 \\ 0 & 6 & 17 & -2 & 0 & 1 \end{array}\right)$$

$$\downarrow \quad r_3 \to r_3 - 2r_2$$

$$\left(\begin{array}{ccc|ccc} 2 & 6 & -3 & 1 & 0 & 0 \\ 0 & 3 & 9 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{array}\right)$$

Rescale rows

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 3 & -1/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right) \qquad \begin{array}{l} r_1 \to \frac{1}{2} r_1 \\ r_2 \to \frac{1}{3} r_2 \\ r_3 \to -r_3 \end{array}$$

Use 3rd row to clear last col.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/2 & 3 & -3/2 \\ 0 & 1 & 0 & -1/3 & -17/3 & 3 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right) \qquad \begin{array}{l} r_2 \to r_2 - 3r_3 \\ r_1 = r_1 + \frac{3}{2} r_3 \end{array}$$

$$\downarrow \quad r_1 \to r_1 - 3r_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 20 & -21/2 \\ 0 & 1 & 0 & -1/3 & -17/3 & 3 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right) = \left(I \mid A^{-1} \right)$$

HW (don't turn in) $\boxed{\text{IMPORTANT}}$

1) Write a program to row-reduce an $m \times n$ matrix to REF

2) Write a program to find inverse of an $n \times n$ matrix.

Forward phase:

Loop on row $j = 1$ to $n-1$

Loop on row $i = j+1$ to n

Subtract multiple of row j from row i

(same as multiplying by

$$E_{ij} = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & 0 & \\ 0 & 0 & 1 & & \\ 0 & -\ell_{ij} & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Result is matrix U in REF (if no swaps are needed)

For 3×3

$U = E_{32} E_{31} E_{21} A$ | $A = \underbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}_{L} U$

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 6 & -3 \\ 2 & 9 & 6 \\ 4 & 18 & 11 \end{pmatrix} \xrightarrow{l_{21}=1} \begin{pmatrix} 2 & 6 & -3 \\ 0 & 3 & 9 \\ 4 & 18 & 11 \end{pmatrix} \xrightarrow{l_{31}=2} \begin{pmatrix} 2 & 6 & -3 \\ 0 & 3 & 9 \\ 0 & 6 & 17 \end{pmatrix}$$

$$\xrightarrow{l_{32}=2} \begin{pmatrix} 2 & 6 & -3 \\ 0 & 3 & 9 \\ 0 & 0 & -1 \end{pmatrix} = U$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & -3 \\ 0 & 3 & 9 \\ 0 & 0 & -1 \end{pmatrix}$$

Backwards phase:

First rescale rows so all pivots $= 1$.

Subtract multiples of last row from previous rows to make last column $= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Subtract multiples of next-to-last row from previous rows to make next-to-last column $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Lather Rinse repeat, (work bottom $\to$ top, right $\to$ left)

Row reduction (forward stage) gives a factorization $A = LU$

So What?

To solve $Ax = b$, first solve $L\vec{c} = \vec{b}$, then solve $A\vec{x} = \vec{c}$.

$$L\underbrace{U\vec{x}}_{\vec{c}} = b$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}}_{L} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$c_1 = b_1$$
$$l_{21}c_1 + c_2 = b_2$$
$$l_{31}c_1 + l_{32}c_2 + c_3 = b_3$$

$$c_1 = b_1$$
$$c_2 = b_2 - l_{21}c_1$$
$$c_3 = b_3 - l_{31}c_1 - l_{32}c_2$$

Get $\vec{c}$ from $\vec{b}$ by forward substitution.

Then solve $U\vec{x} = \vec{c}$

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$U_{33} X_3 = C_3 \implies X_3 = C_3 / U_{33}$$

$$U_{22} X_2 + U_{23} X_3 = C_2 \implies X_2 = (C_2 - U_{23} X_3) / U_{22}$$

$$\vdots$$

Solve by back-substitution.

One (hard) square problem

= two (easy) triangular problems.

$$A = \begin{pmatrix} 2 & 6 & -3 \\ 2 & 9 & 6 \\ 4 & 18 & 11 \end{pmatrix} = LU$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 6 & -3 \\ 0 & 3 & 9 \\ 0 & 0 & -1 \end{pmatrix}}_{U}$$

$$A\vec{x} = \begin{pmatrix} 2 \\ 23 \\ 44 \end{pmatrix}$$

$$L\vec{c} = \begin{pmatrix} 2 \\ 23 \\ 44 \end{pmatrix} \qquad U\vec{x} = \vec{c}$$

$$\left.\begin{aligned} c_1 &= 2 \\ c_1 + c_2 &= 23 \\ 2c_1 + 2c_2 + c_3 &= 44 \end{aligned}\right\} \Longrightarrow \begin{aligned} c_1 &= 2 \\ c_2 &= 21 \\ c_3 &= 44 - 2c_1 - 2c_2 = -2 \end{aligned}$$

$$\left.\begin{aligned} 2x_1 + 6x_2 - 3x_3 &= 2 \\ 3x_2 + 9x_3 &= 21 \\ -x_3 &= -2 \end{aligned}\right\} \qquad \begin{aligned} x_1 &= (2 - 6x_2 + 3x_3)/2 = 1 \\ x_2 &= (21 - 9(2))/3 = 1 \\ x_3 &= 2 \end{aligned}$$

$(U\vec{x} = c)$

$$A = \underbrace{\begin{pmatrix} 1 & & \\ \ell & 1 & \\ \ell & \ell & 1 \end{pmatrix}}_{L} \begin{pmatrix} U_{11} & \sim & \sim \\ 0 & U_{22} & \sim \\ 0 & 0 & U_{33} \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & 0 & 0 \\ 0 & U_{22} & 0 \\ 0 & 0 & U_{33} \end{pmatrix} \begin{pmatrix} 1 & \sim & \sim \\ 0 & 1 & \sim \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \quad L \qquad D \qquad U$$

This assumed

1) A invertible.
2) No row swaps needed

(PLU or PLDU) P = permutation.

If A not invertible,

$$U = \begin{pmatrix} 1 & \sim & \sim \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \sim & \sim \\ 0 & 1 & \sim \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & \sim \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$