Solving $Ax=0$

1) Put A in <u>reduced</u> row-echelon form

$$R = \mathrm{rref}(A)$$

2) Each (nonzero) row gives a pivot variable in terms of the free variables.

3) Add trivial equations $X_{free} = X_{free}$

$$\vec{X} = \begin{pmatrix} \text{Something} \\ \text{in terms} \\ \text{of free} \end{pmatrix}$$

4) Special solutions have $\begin{cases} \text{one free} = 1 \\ \text{others} = 0 \end{cases}$

5) $N = \begin{pmatrix} \text{Special} \\ \text{Solutions} \end{pmatrix}$ = Null-matrix of A.

6) General solution to $Ax=0$ is
= arbitrary linear comb of special soln
= $\mathrm{Col}(N)$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 5 & 8 \\ 3 & 3 & 4 & 6 & 9 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 3 & 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

REF

$$\begin{pmatrix} 1 & 1 & 0 & -2 & -5 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

x_1, x_3 pivot variables

x_2, x_4, x_5 free

$x_1 + x_2 - 2x_4 - 5x_5 = 0$

$x_3 + 3x_4 + 6x_5 = 0$

$$\begin{aligned} x_1 &= -x_2 + 2x_4 + 5x_5 \\ x_2 &= x_2 \\ x_3 &= -3x_4 - 6x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}$$

Special soln

$$\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\vec{s}_2}, \underbrace{\begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}}_{\vec{s}_4}, \underbrace{\begin{pmatrix} 5 \\ 0 \\ -6 \\ 0 \\ 1 \end{pmatrix}}_{\vec{s}_5}$$

$$N = \begin{pmatrix} -1 & 2 & 5 \\ 1 & 0 & 0 \\ 0 & -3 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\vec{s}_2 \ \vec{s}_4 \ \vec{s}_5)$$

$$\vec{x} = x_2 \vec{s}_2 + x_4 \vec{s}_4 + x_5 \vec{s}_5$$

Note: Each special soln gives a ~~pivo~~ free column in terms of ~~a~~ the pivot columns.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 5 & 8 \\ 3 & 3 & 4 & 6 & 9 \end{pmatrix} = \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{pmatrix}$$

$$A\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \qquad -\vec{a}_1 + \vec{a}_2 = 0 \quad \Rightarrow \vec{a}_2 = \vec{a}_1$$

$$A\begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} = 0 \qquad 2\vec{a}_1 - 3\vec{a}_3 + \vec{a}_4 = 0 \quad \Rightarrow \vec{a}_4 = -2\vec{a}_1 + 3\vec{a}_3$$

$$A\begin{pmatrix} 5 \\ 0 \\ -6 \\ 0 \\ 1 \end{pmatrix} = 0 \qquad 5\vec{a}_1 - 6\vec{a}_3 + \vec{a}_5 = 0 \qquad \vec{a}_5 = -5\vec{a}_1 + 6\vec{a}_3$$

Free columns are redundant;

$$\text{Col}(A) = \text{Span(pivot columns of } A)$$
$$\neq \text{Span(pivot columns of } R)$$

Pivot columns are linearly independent.

The rref of a matrix is <u>unique</u>.

R determines solutions to $Ax=0$

Solutions to $Ax=0$ determine R.

We can speak of rref(A).

If A is an invertible $n \times n$ matrix,

rref(A) $= I$

If A is a singular $n \times n$ matrix,
rref (A) has $< n$ pivots.

Solving $Ax=b$

1) Reduce $(A \mid b)$ to $(R \mid \vec{d})$
2) Check for contradictions, if none....
3) Find one solution $\vec{x}_p$ "particular solution"

   Easiest way: set all free $=0$.

4) If $\vec{x}_n \in Nul(A)$, $A(\vec{x}_n+\vec{x}_p)=0+\vec{b}=\vec{b}$

   If $\cancel{x}\ A\vec{x}=b$, $x=(\vec{x}-\vec{x}_p)+\vec{x}_p$

   $A(\vec{x}-\vec{x}_p)=A\vec{x}-A\vec{x}_p=\vec{b}-\vec{b}=0$, so $\vec{x}-\vec{x}_p \in Nul(A)$.

$$\left(\text{Solutions to } Ax=b\right)=\vec{x}_p+Nul(A)$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 5 & 8 \\ 3 & 3 & 4 & 6 & 9 \end{pmatrix}}_{A} \underbrace{\left|\begin{matrix} 2 \\ 3 \\ 5 \end{matrix}\right.}_{b} \rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 & 6 & -1 \\ 0 & 0 & 1 & 3 & 6 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 & 6 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & -2 & -5 & 3 \\ 0 & 0 & 1 & 3 & 6 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} x_1 &= 3 - x_2 + 2x_4 + 5x_5 \\ x_2 &= x_2 \\ x_3 &= -1 - 3x_4 - 6x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}$$

$$\vec{x} = \underbrace{\begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{x}_p} + \underbrace{x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 5 \\ 0 \\ -6 \\ 0 \\ 1 \end{pmatrix}}_{Nul(A)}$$

$$\left(\begin{array}{ccccc|c} 1&1&1&1&1&2\\ 2&2&3&5&8&3\\ 3&3&4&6&9&6 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1&1&1&1&1&2\\ 0&0&1&3&6&-1\\ 0&0&1&3&6&0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1&1&1&1&1&2\\ 0&0&1&3&6&-1\\ 0&0&0&0&0&1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccccc|c} 1&1&0&-2&-5&3\\ 0&0&1&3&6&-1\\ 0&0&0&0&0&1 \end{array}\right)$$

You get a solution only when entries of $\vec{d}$ that match zero rows vanish.

$$\left(\begin{array}{ccccc|c} 1&1&1&1&1&b_1\\ 2&2&3&5&8&b_2\\ 3&3&4&6&9&b_3 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1&1&1&1&1&b_1\\ 0&0&1&3&6&b_2-2b_1\\ 0&0&1&3&6&b_3-3b_1 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1&1&1&1&1&b_1\\ 0&0&1&3&6&b_2-2b_1\\ 0&0&0&0&0&b_3-b_1-b_2 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccccc|c} 1&1&0&-2&-5&3b_1-b_2\\ 0&0&1&3&6&b_2-2b_1\\ 0&0&0&0&0&b_3-b_1-b_2 \end{array}\right)$$

There is a solution if and only if

$$b_3 = b_1 + b_2$$

The rank of a matrix is # of pivots.

~~###~~ Let A be an $m \times n$ matrix of rank K.

If $K < m$, contradictions are possible in $Ax = b$.
$m - K$ rows of 0's in rref(A)

If $K = m$, no contradictions, $Col(A) = \mathbb{R}^m$.
Columns of A span $\mathbb{R}^m$.

If $K < n$ there are free variables.
$Nul(A) \neq 0$
Columns are linearly dependent.
Free columns are lin. comb. of pivot cols.

If $K = n$. No free variables
$Nul(A) = 0$
Columns are linearly independent.

~~Im~~ If $m < n$ (short & fat) $K \leq m < n$, so
columns are linearly dependent.
Can't have more than m lin. ind. vectors in $\mathbb{R}^m$.

If $m > n$ (tall & skinny) $K \leq n < m$, so
columns don't span $\mathbb{R}^m$.
Need at least m vectors to span $\mathbb{R}^m$.

If $m=n$, then either

i) $K<m$, so columns lin dependent and don't span, or

ii) $K=m$, so columns lin independent and span.

a Given : a vector space V (think of $\mathbb{R}^m$)

and a collection $\mathcal{B}=\{\vec{v}_1,\dots \vec{v}_n\}$ of vectors in V.

<u>Def</u> $\mathcal{B}$ is linearly independent if the ~~al~~ only way to get $a_1\vec{v}_1+\cdots+a_n\vec{v}_n=0$ is with $a_1=a_2=\cdots=a_n=0$.

Otherwise, $\mathcal{B}$ is linearly <u>dependent</u>.

<u>Thm</u> If $\mathcal{B}$ is lin dependent, ~~you~~ one of $\vec{v}_1,\dots\vec{v}_n$ is a lin comb of previous ones.

Linearly dependent $\Longleftrightarrow$ redundant.

<u>Def</u> $\mathcal{B}$ <u>spans V</u> if $\text{Span}(\mathcal{B})=V$.

<u>Def</u> $\mathcal{B}$ is a <u>basis</u> for V if $\mathcal{B}$ is linearly independent and spans.

<u>Fact</u> Every basis for $\mathbb{R}^m$ has exactly m vectors in it.

<u>Def</u> The dimension of V = # of vectors in a basis.

Basis for Col(A) = {pivot columns}

Basis for Nul(A) = {special solutions}

~~Basis for~~