$$X(n) = 3\,X(n-1) \implies X(n) = 3^n X(0)$$

$$\frac{dx}{dt} = 3x \implies X(t) = e^{3t} X(0)$$

$$\frac{d^2x}{dt^2} = -3x \implies X(t) = X(0)\cos(\sqrt{3}\,t) + \frac{\dot{x}(0)}{\sqrt{3}}\sin(\sqrt{3}\,t)$$

$$X_1(n) = 2X_1(n-1) + X_2(n-1)$$
$$X_2(n) = X_1(n-1) + 2X_2(n-1)$$

$X_1(0) = 5$

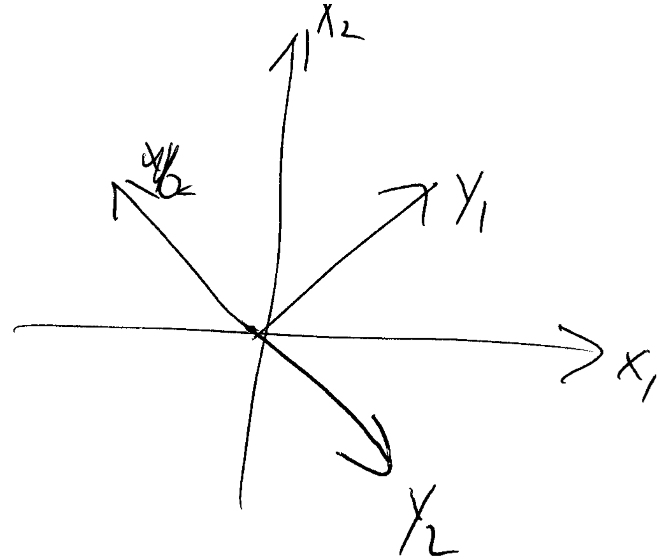
$X_2(0) = 7$

$$\vec{X}(n) = A\,\vec{x}(n-1) \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\vec{x}(n) = A^n\,\vec{x}(0)$$

define $y_1 = (x_1 + x_2)$

$y_2 = x_1 - x_2$



$$x_1 = \frac{y_1 + y_2}{2}$$

$$x_2 = \frac{y_1 - y_2}{2}$$

$$x_1(n) + x_2(n) = 3x_1(n-1) + 3x_2(n-1)$$

$$y_1(n) = 3y_1(n-1)$$

$$x_1(n) - x_2(n) = x_1(n-1) - x_2(n-1)$$

$$y_2(n) = y_2(n-1)$$

$$\begin{tikzcd}
\begin{pmatrix}12\\-2\end{pmatrix} = \vec{y}(0) \arrow[r, "\begin{pmatrix}3^n & 0\\ 0 & 1^n\end{pmatrix}", "\text{easy}"'] & \vec{y}(n) = \begin{pmatrix}12\cdot 3^n\\ -2\end{pmatrix} \arrow[d] \\
\begin{pmatrix}5\\7\end{pmatrix} = \vec{X}(0) \arrow[u] \arrow[r, "A^n\ (\text{hard})"] & \vec{X}(n)
\end{tikzcd}$$

$$\vec{X}(n) = \begin{pmatrix}(12\cdot 3^n - 2)/2\\ (12\cdot 3^n + 2)/2\end{pmatrix} = \begin{pmatrix}6\cdot 3^n - 1\\ 6\cdot 3^n + 1\end{pmatrix}$$

$$\frac{dx_1}{dt} = 2x_1 + x_2 \qquad \frac{dy_1}{dt} = 3y_1 \Rightarrow y_1(t) = y_1(0)e^{3t}$$

$$\frac{dx_2}{dt} = x_1 + 2x_2 \qquad \frac{dy_2}{dt} = y_2 \Rightarrow y_2(t) = y_2(0)e^{t}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

3 and 1 are eigenvalues of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Def

If A is a square matrix, $\vec{x} \neq \vec{0}$

and $A\vec{x} = \lambda \vec{x}$ $(\lambda = \text{scalar})$

we say $\vec{x}$ is an eigenvector of A with

eigen value λ.

Goal: Find a basis of $\mathbb{R}^n$ consisting of e-vecs of A.

$Ax = \lambda x = \lambda I \vec{x}$

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \vec{x}$

$(A - \lambda I)\vec{x} = 0$

$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \vec{x} = 0$

$(A - \lambda I)$ is singular

$\det(A - \lambda I) = 0$

$(2-\lambda)(2-\lambda) - 1 \cdot 1 = 0$

$\lambda^2 - 4\lambda + 4 - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 1)(\lambda - 3) = 0 \Rightarrow \lambda = 1 \text{ or } 3$

$\lambda = \dfrac{4 \pm \sqrt{16-12}}{2} = \dfrac{4 \pm \sqrt{4}}{2} =$

Fact: Eigenvalues of A = roots of $\det(A - \lambda I)$

Eigenvector $\in$ Nul $(A - \lambda I)$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$\lambda = 1$: $\quad A - \lambda I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

row reduce to $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\qquad$ $x_1 = -x_2$
$x_2 = x_2$

$$\vec{x} = c \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\lambda = 3$: $\quad A - \lambda I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} +1 & -1 \\ 0 & 0 \end{pmatrix}$ $\qquad$ $x_1 = x_2$
$x_2 = x_2$

$$\vec{x} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)(2-\lambda) - 2$$

$$= \lambda^2 - 5\lambda + 4$$

$$\lambda = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2}$$

$$= 4, 1$$

Characteristic poly of $A = \det(\lambda I - A) = p_A(\lambda)$
$= \pm \det(A - \lambda I)$

E-vals are roots of $p_A(\lambda)$

$\lambda = 1$: $A - \lambda I = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix}$

$x_1 = -\frac{1}{2} x_2$
$x_2 = x_2$

$\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$\lambda = 4$. $\quad A - \lambda I = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Get e-vecs by row-reducing $A - \lambda \underline{I}$.