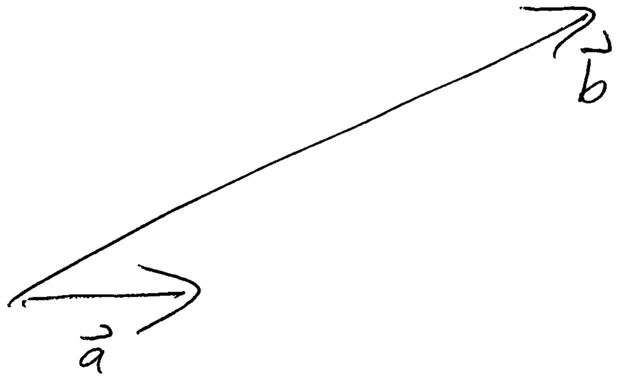


$$P_{\vec{a}}\,\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a} = \hat{x}\,\vec{a}$$

If $\vec{a}$ is a unit vector, $\vec{a}\cdot\vec{a}=1$,

$$P_{\vec{a}}\,\vec{b} = (\vec{a}\cdot\vec{b})\,\vec{a} = \hat{x}\,\vec{a} \qquad \hat{x} = \vec{a}\cdot\vec{b}$$

$$P_{\vec{a}} = \vec{a}\,\vec{a}^T$$

If $A = (\vec{a}_1 \cdots \vec{a}_n)$ $\quad V = \mathrm{Col}(A) = \mathrm{Span}\{\vec{a}_1, \cdots \vec{a}_n\}$.

To get $P_V\vec{b}$; solve $A^TA\hat{x} = A^T\vec{b}$, then

$$P_V\vec{b} = A\hat{x}$$

If columns of A are lin ind, A^TA is invertible,

$$\hat{x} = (A^TA)^{-1}A^T\vec{b}\,, \qquad P_V\vec{b} = A(A^TA)^{-1}A^T\vec{b}.$$

$$P_V = A(A^TA)^{-1}A^T$$

Thm If columns of A are $\perp$, then $P_V = P_{\vec{a}_1} + P_{\vec{a}_2} + \cdots + P_{\vec{a}_n}$.

Restatement: If $\vec{a}_1, \ldots \vec{a}_n$ are orthogonal and b is any vector, then

$$\vec{b} = \left(\frac{\vec{a}_1 \cdot \vec{b}}{\vec{a}_1 \cdot \vec{a}_1}\right) \vec{a}_1 + \cdots + \frac{(\vec{a}_n \cdot \vec{b})}{(\vec{a}_n \cdot \vec{a}_n)} \vec{a}_n + \vec{b}_\perp,$$

where $b_\perp \in V^\perp$.

Special case: If $\vec{a}_1, \ldots \vec{a}_n$ are **orthonormal**, then

$$\vec{b} = (\vec{a}_1 \cdot \vec{b})\vec{a}_1 + \cdots (\vec{a}_n \cdot \vec{b})\vec{a}_n + \vec{b}_\perp.$$

$$P_V = \vec{a}_1 \vec{a}_1^T + \vec{a}_2 \vec{a}_2^T + \cdots + \vec{a}_n \vec{a}_n^T$$

Counterexamples.

In $\mathbb{R}^2$, $\vec{a}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$\vec{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$\frac{\vec{a}_1 \cdot \vec{b}}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 = \frac{5}{5} \vec{a}_1 = \vec{a}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\frac{\vec{a}_2 \cdot \vec{b}}{\vec{a}_2 \cdot \vec{a}_2} = \frac{4}{5} \vec{a}_2 = \begin{pmatrix} 4/5 \\ 8/5 \end{pmatrix}$$

$P_V \neq P_{\vec{a}_1} + P_{\vec{a}_2}$ since $\vec{a}_1$ and $\vec{a}_2$ are not $\perp$

In $\mathbb{R}^2$, $\vec{a}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. $\vec{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$$\vec{b} = \frac{6}{4} \vec{a}_1 + \frac{10}{4} \vec{a}_2 \neq 6\vec{a}_1 + 10\vec{a}_2$$

$\vec{a}_1 \cdot \vec{a}_1$ $\qquad$ $\vec{a}_2 \cdot \vec{a}_2$

Proof of thm

$$\vec{b} = \vec{b}_{\parallel} + \vec{b}_{\perp}$$

$$= c_1 \vec{a}_1 + \cdots + c_n \vec{a}_n + \vec{b}_{\perp}$$

$$\vec{a}_1 \cdot \vec{b} = c_1 \vec{a}_1 \cdot \vec{a}_1 + c_2 \vec{a}_1 \cdot \vec{a}_2 + \cdots + c_n \vec{a}_1 \cdot \vec{a}_n + \vec{a}_1 \cdot \vec{b}_{\perp}$$

$$= c_1 \vec{a}_1 \cdot \vec{a}_1$$

$$\Rightarrow c_1 = \frac{\vec{a}_1 \cdot \vec{b}}{\vec{a}_1 \cdot \vec{a}_1}, \qquad \vec{c}_k = \frac{\vec{a}_k \cdot \vec{b}}{\vec{a}_k \cdot \vec{a}_k}$$

$$\vec{b} = \left(\frac{\vec{a}_1 \cdot \vec{b}}{\vec{a}_1 \cdot \vec{a}_1}\right) \vec{a}_1 + \cdots + \left(\frac{\vec{a}_n \cdot \vec{b}}{\vec{a}_n \cdot \vec{a}_n}\right) \vec{a}_n + \vec{b}_{\perp}$$

$$A = \begin{pmatrix} 1 & 2 & 6 \\ 1 & 4 & 0 \\ 1 & 6 & -2 \\ 1 & 8 & 0 \end{pmatrix} = \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{pmatrix}$$

Find an orthogonal basis for Col(A).
$(\vec{v}_1, \dots \vec{v}_n)$

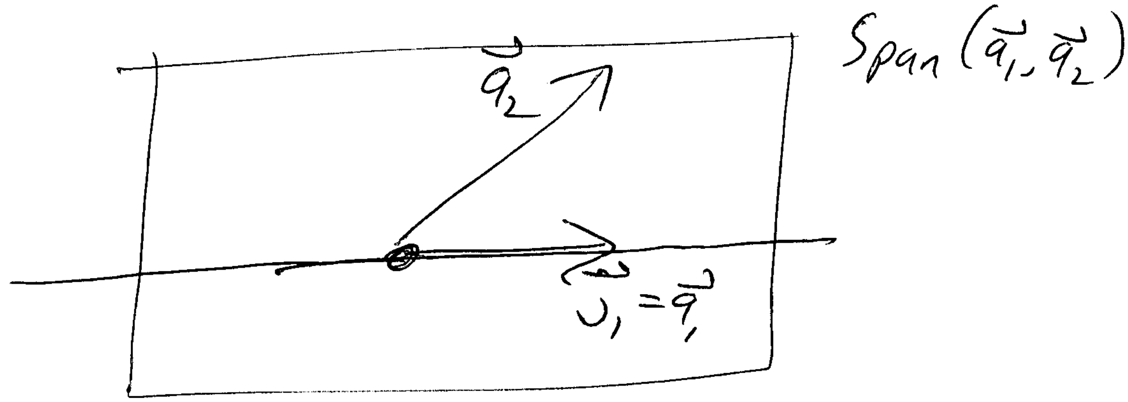
Find an orthonormal " " Col(A).
$(\vec{q}_1, \dots \vec{q}_n)$

Gram- Schmidt (sp?) process

Idea: For each $k = 1, \dots n$, we want $(\vec{v}_1, \dots \vec{v}_k)$ to be an orthogonal basis for $\text{Span}(\vec{a}_1, \dots \vec{a}_k)$

Step 1: $\vec{v}_1 = \vec{a}_1$.

Step 2:

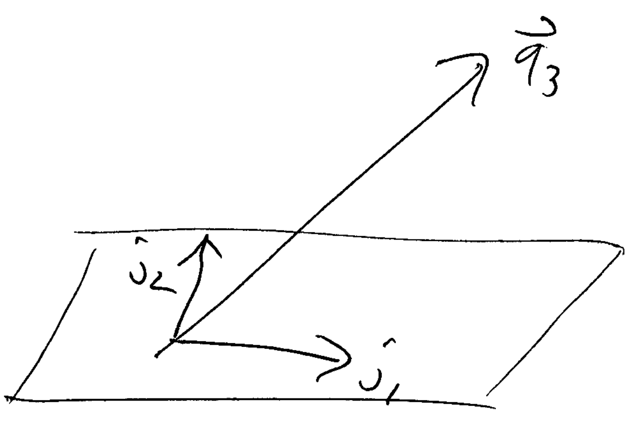


$$\vec{v}_2 = (\vec{a}_2)_\perp = \text{part of } \vec{a}_2 \perp \text{ to } \vec{v}_1.$$

$$= \vec{a}_2 - (P_{\vec{v}_1} \vec{a}_2)$$

$$= \vec{a}_2 - \left(\frac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1$$

Step 3



$$\vec{v}_3 = (\vec{a}_3)_\perp = \text{part of } \vec{a}_3 \perp \text{ to } Span\{\vec{v}_1, \vec{v}_2\}$$

$$= \vec{a}_3 - P_{\vec{v}_1} \vec{a}_3 - P_{\vec{v}_2} \vec{a}_3$$

Step 4

$$\cancel{\vec{a}_4} \; \vec{v}_4 = \vec{a}_4 - P_{\vec{v}_1}\vec{a}_4 - P_{\vec{v}_2}\vec{a}_4 - P_{\vec{v}_3}\vec{a}_4$$

Step k

$$\vec{v}_k = \vec{a}_k - \underbrace{P_{\vec{v}_1}\vec{a}_k - \dots - P_{\vec{v}_{k-1}}\vec{a}_k}_{(\vec{a}_k)_{\parallel}}$$

$$\boxed{\text{NOT} \quad \vec{a}_k - \sum_{i=1}^{k-1} P_{\vec{a}_i}\vec{a}_k}$$

Example

$$A = \begin{pmatrix} 1 & 2 & 6 \\ 1 & 4 & 0 \\ 1 & 6 & -2 \\ 1 & 8 & 0 \end{pmatrix}$$

$$\vec{v}_1 = \vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_2 = \vec{a}_2 - P_{\vec{v}_1} \vec{a}_2 = \vec{a}_2 - \frac{20}{4} \vec{v}_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix}$$

$$\vec{v}_3 = \vec{a}_3 - P_{\vec{v}_1} \vec{a}_3 - P_{\vec{v}_2} \vec{a}_3 = \vec{a}_3 - \left(\frac{\vec{v}_1 \cdot \vec{a}_3}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 - \left(\frac{\vec{v}_2 \cdot \vec{a}_3}{\vec{v}_2 \cdot \vec{v}_2} \right) \vec{v}_2$$

$$= \vec{a}_3 - \frac{4}{4} \vec{v}_1 - \frac{(-20)}{20} \vec{v}_2 = \begin{pmatrix} 6 \\ 0 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \\ 2 \end{pmatrix}.$$

$$\vec{q}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{\vec{v}_1}{\sqrt{\vec{v}_1 \cdot \vec{v}_1}} = \frac{1}{\sqrt{4}} \vec{v}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$\vec{q}_2 = \frac{\tilde{\vec{v}}_2}{|\tilde{\vec{v}}_2|} = \frac{1}{2\sqrt{5}} \begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix}$$

$$\vec{q}_3 = \frac{\tilde{\vec{v}}_3}{|\tilde{\vec{v}}_3|} = \frac{1}{4} \tilde{\vec{v}}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$\operatorname{Span}\{\vec{q}_1, \vec{q}_2, \vec{q}_3\} = \operatorname{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}.$$

For any $\vec{b} \in \operatorname{Span}(\vec{a}_1, \vec{a}_2, \vec{a}_3,)$

$$\vec{b} = (\vec{b}\cdot\vec{q}_1)\vec{q}_1 + (\vec{b}\cdot\vec{q}_2)\vec{q}_2 + (\vec{b}\cdot\vec{q}_3)\vec{q}_3.$$

$$\vec{a}_1 = (\vec{a}_1\cdot\vec{q}_1)\vec{q}_1$$

$$\vec{a}_2 = (\vec{a}_2\cdot\vec{q}_1)\vec{q}_1 + (\vec{a}_2\cdot\vec{q}_2)\vec{q}_2$$

$$\vec{a}_3 = (\vec{a}_3\cdot\vec{q}_1)\vec{q}_1 + (\vec{a}_3\cdot\vec{q}_2)\vec{q}_2 + (\vec{a}_3\cdot\vec{q}_3)\vec{q}_3$$

$$A = \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{pmatrix} \qquad Q = \begin{pmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{pmatrix}$$

$$A = QR$$

$$R = \begin{pmatrix} (\vec{a}_1\cdot\vec{q}_1) & \vec{a}_2\cdot\vec{q}_1 & \vec{a}_3\cdot\vec{q}_1 \\ 0 & (\vec{a}_2\cdot\vec{q}_2) & \vec{a}_3\cdot\vec{q}_2 \\ 0 & 0 & \vec{a}_3\cdot\vec{q}_3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 6 \\ 1 & 4 & 0 \\ 1 & 6 & -2 \\ 1 & 8 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 1/2 & \frac{-3}{2\sqrt{5}} & 1/2 \\ 1/2 & -\frac{1}{2\sqrt{5}} & -1/2 \\ 1/2 & 1/2\sqrt{5} & -1/2 \\ 1/2 & 3/2\sqrt{5} & 1/2 \end{pmatrix}$$

$$\vec{a}_1 \cdot \vec{q}_1 = 2$$

$$\vec{a}_2 \cdot \vec{q}_1 = 10$$

$$\vec{a}_2 \cdot \vec{q}_2 = \frac{20}{2\sqrt{5}} = 2\sqrt{5}$$

$$\vec{a}_3 \cdot \vec{q}_1 = 2$$

$$\vec{a}_3 \cdot \vec{q}_2 = -2\sqrt{5}$$

$$\vec{a}_3 \cdot \vec{q}_3 = 4$$

$$\begin{array}{cc} & n \\ m & \begin{pmatrix} 1 & 2 & 6 \\ 1 & 4 & 0 \\ 1 & 6 & -2 \\ 1 & 8 & 0 \end{pmatrix} \end{array} = \begin{array}{cc} & n \\ m & \begin{pmatrix} 1/2 & \frac{-3}{2\sqrt{5}} & 1/2 \\ 1/2 & -\frac{1}{2\sqrt{5}} & -1/2 \\ 1/2 & 1/2\sqrt{5} & -1/2 \\ 1/2 & 3/2\sqrt{5} & 1/2 \end{pmatrix} \end{array} \begin{array}{cc} & n \\ n & \begin{pmatrix} 2 & 10 & 2 \\ 0 & 2\sqrt{5} & -2\sqrt{5} \\ 0 & 0 & 4 \end{pmatrix} \end{array}$$

$$Q^T Q = \begin{pmatrix} q_1^T \\ q_2^T \\ q_3^T \end{pmatrix} \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}$$

$$= \begin{pmatrix} q_1 \cdot q_1 & q_1 \cdot q_2 & q_1 \cdot q_3 \\ q_2 \cdot q_1 & \ddots & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^T A \vec{\hat{x}} = A^T \vec{b}$$

$$R^T \cancel{Q^T Q} R \hat{x} = R^T Q^T b$$

$$R^T R \hat{x} = R^T Q^T b$$

$$\boxed{R \hat{x} = Q^T b}$$