Exam Thursday

6.1 – ~~6.4~~ 6.3, 8.3

Usual ground rules.

Extra office hrs tomorrow

10 – 12:30 RLM 9.114

$$\begin{pmatrix} 0 & 9 \\ 1 & 0 \end{pmatrix} \qquad \mathrm{Tr} = 0 \qquad \lambda = 3, -3$$

$$\det = -9$$

$$A - \lambda I$$

$$E_3: \begin{pmatrix} -3 & 9 \\ 1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_1 &= 3x_2 \\ x_2 &= x_2 \end{aligned} \qquad \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$E_{-3}: \begin{pmatrix} 3 & 9 \\ 1 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 9 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 1 & 0 \end{pmatrix} + 2I \qquad \lambda = 5, -1$$

e-vecs same

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad$ $Tr = 0$ $\quad$ $\lambda = \pm i$

$det = 1$

$S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ $\quad$ $\lambda = 2 \pm i$ $\quad$ $S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$

~~Problem~~

Coupled evolution problems

- $\begin{pmatrix} \text{evolution} \\ \text{of } \vec{x} \end{pmatrix} = A\vec{x}$

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(1) $\vec{x}(n+1) = A\vec{x}$ discrete evolution

(2) $\dfrac{d\vec{x}}{dt} = A\vec{x}$ System of ODEs.

(3) $\dfrac{d^2\vec{x}}{dt^2} = A\vec{x}$ Coupled oscillators.

1) Diagonalize $A = S \Lambda S^{-1}$

$$S = (\vec{b}_1, \cdots \vec{b}_m)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

2) Expand $\vec{x}$ in terms of basis $\{\vec{b}_1, \cdots \vec{b}_m\}$

$$\vec{x} = \underbrace{y_1}\, \vec{b}_1 + \cdots \qquad + \underbrace{y_m}\, \vec{b}_m$$

$$\vec{x} = S\vec{y} \qquad\qquad \vec{y} = S^{-1}\vec{x}$$

3) Rewrite equations in terms of $\vec{y}$.
Everything decouples.

(evolution of y_i) $= \lambda_i y_i$

4) Solve one variable at a time

5) Convert back to $\vec{x}$.

$$
\begin{tikzcd}
\vec{y}(0) \arrow[r, "\text{One variable}", "\text{at a time}"'] & \vec{y}(\text{end}) \arrow[d, "S"] \\
\vec{X}(0) \arrow[u, "S^{-1}"] \arrow[r, dashed, "\text{Coupled}", "\text{system}"'] & \vec{X}(\text{end})
\end{tikzcd}
$$

$$X_1(n+1) = 2X_1(n) + 9X_2(n)$$
$$X_2(n+1) = X_1(n) + 2X_2(n)$$
$$\vec{X}(0) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 9 \\ 1 & 2 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \qquad S = \begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix}$$

$$S^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}$$

$$\vec{Y}(0) = \frac{1}{6}\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -11/6 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 5 \\ -2 \end{pmatrix}}_{\vec{X}(0)} = \underbrace{\frac{-1}{6}}_{y_1(0)}\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \underbrace{\frac{11}{6}}_{y_2(0)}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$y_1(n+1) = 5\,y_1(n) \qquad y_1(n) = 5^n y_1(0) = \left(\frac{-1}{6}\right)5^n$$
$$y_2(n+1) = -1\,y_2(n) \qquad y_2(n) = (-1)^n y_2(0) = \frac{-11}{6}(-1)^n$$

$$\vec{X}(n) = y_1(n)\vec{b}_1 + y_2(n)\vec{b}_2 = \frac{-1}{6}\cdot 5^n\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \frac{11}{6}(-1)^n\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$
$$= \frac{1}{6}\begin{pmatrix} -3\cdot(5^n) + 33(-1)^n \\ -(5^n) - 11(-1)^n \end{pmatrix}$$

$$\vec{x} = S\vec{y}$$

$$\vec{x}(n+1) = A\vec{x}(n) = S\Lambda S^{-1}\vec{x}(n)$$

$$S\vec{y}(n+1) = S\Lambda\vec{y}(n)$$

$$\vec{y}(n+1) = \Lambda\vec{y}(n)$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}(n+1) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}(n) = \begin{pmatrix} \lambda_1 y_1(n) \\ \vdots \\ \lambda_m y_m(n) \end{pmatrix}$$

If $\vec{X}(0)$ unknown,

$y_1(0) = c_1$ $\qquad$ $y_1(n) = c_1 \lambda_1^n$

$y_2(0) = c_2$ $\qquad$ $y_2(n) = c_2 \lambda_2^n$

$\vdots$ $\qquad$ $\vdots$

$$\boxed{\vec{X}(n) = c_1 \lambda_1^n \vec{b}_1 + c_2 \lambda_2^n \vec{b}_2 + \cdots + c_m \lambda_m^n \vec{b}_m}$$

Terms with $|\lambda_i| > 1$ grow exponentially (unstable mode)

Terms with $|\lambda_i| < 1$ shrink exponentially (stable)

Terms with $|\lambda_i| = 1$ neither grow nor shrink.

Eventually $\vec{X}$ points in direction of e-vec w/ biggest $|\lambda_i|$ (dominant e-val, e-vec)

$$\frac{dx_1}{dt} = 2x_1 + 9x_2 \qquad \vec{x}(0) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$\frac{dx_2}{dt} = x_1 + 2x_2$$

$$\begin{tikzcd}
\begin{pmatrix} -1/6 \\ -11/6 \end{pmatrix} = y(0) \arrow[rr] & & y(t) \arrow[d] \\
\begin{pmatrix} 5 \\ -2 \end{pmatrix} = X(0) \arrow[u] & & X(t)
\end{tikzcd}$$

$$\frac{dy_1}{dt} = 5y_1 \qquad y_1(t) = e^{5t} y_1(0)$$

$$\frac{dy_2}{dt} = -1\,y_2 \qquad y_2(t) = e^{-t} y_2(0)$$

$$\vec{y}(t) = \begin{pmatrix} -e^{5t}/6 \\ -11\,e^{-t}/6 \end{pmatrix} \qquad \vec{x}(t) = -\frac{e^{5t}}{6}\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \frac{11\,e^{-t}}{6}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

General solution to $\frac{d\vec{x}}{dt} = A\vec{x}$

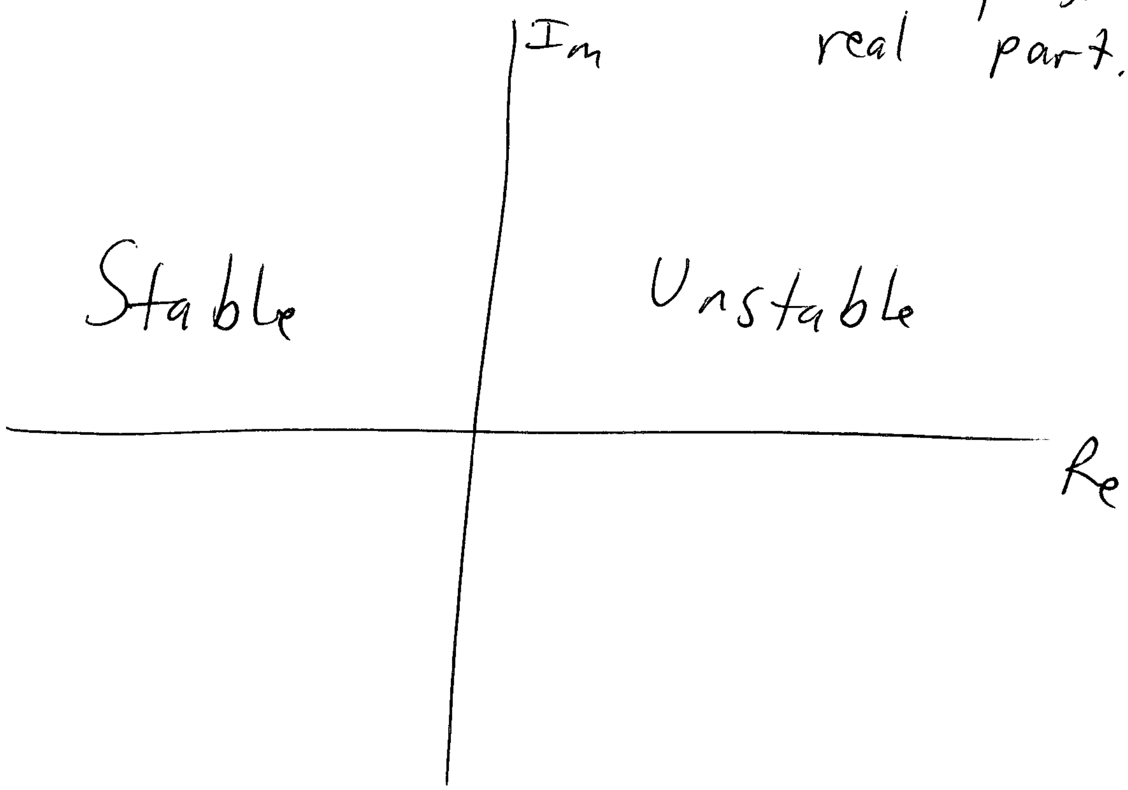
$$\vec{X}(t) = \sum_{j=1}^{m} c_j e^{\lambda_j t} \vec{b}_j$$

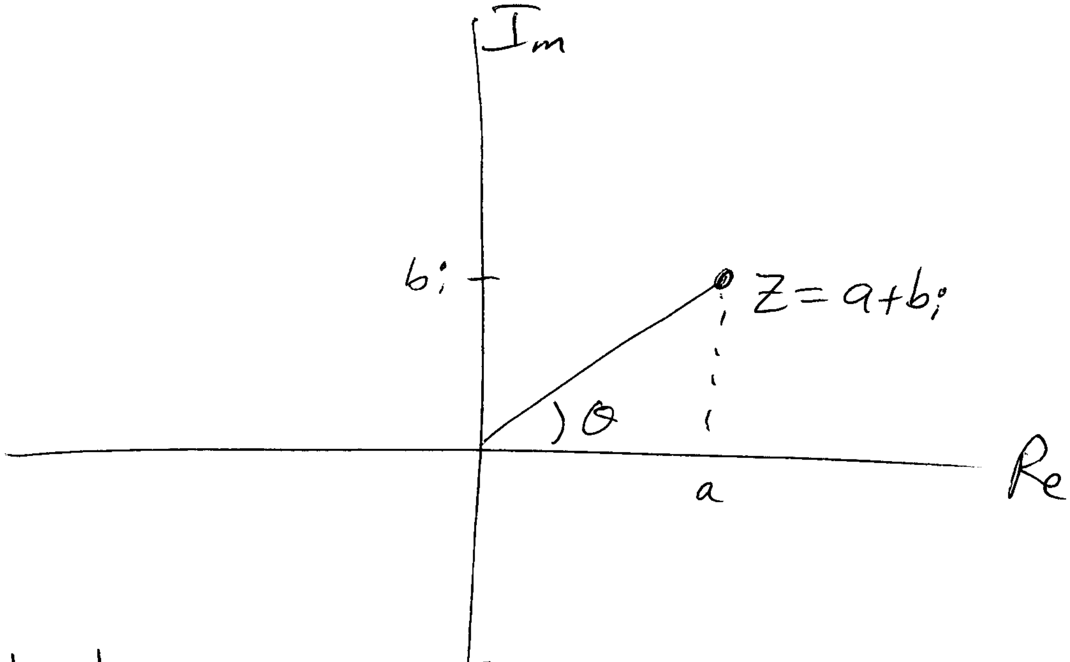
Terms with $Re(\lambda) > 0$ grow

$Re(\lambda) < 0$ shrink

$Re(\lambda) = 0$ stick around.

"dominant" means "most positive" real part.





$$|z| = \sqrt{a^2+b^2}$$

$$\operatorname{Arg}(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$|zw| = |z|\,|w|$$

$$\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$$

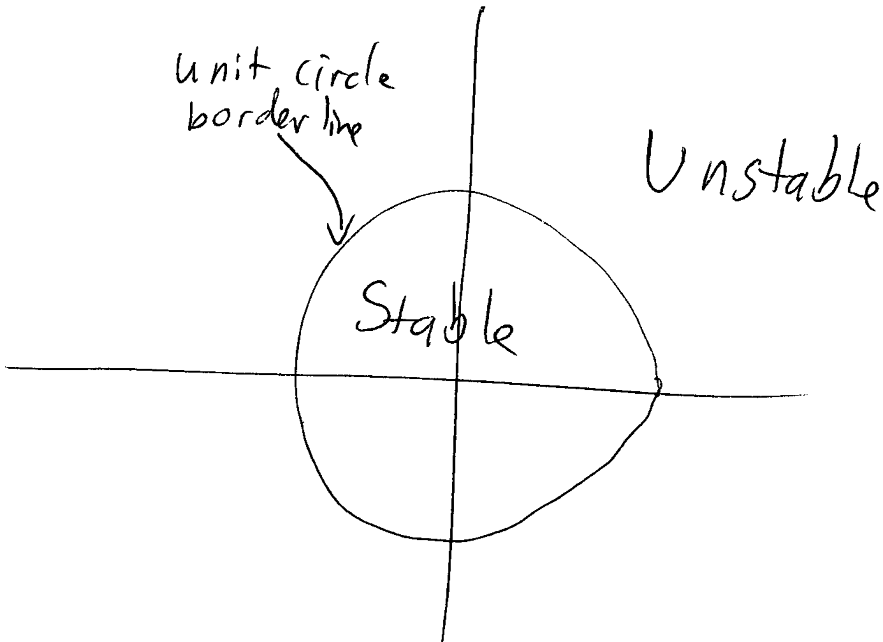
$$|z^n| = |z|^n$$

$$\operatorname{Arg}(z^n) = n\operatorname{Arg}(z)$$

When dealing with $\vec{x}(n+1) = A\vec{x}(n)$ problems w/ complex eigenvalues,

$|\lambda|$ gives growth rate.

Dominant e-val is e-val w/ biggest $|\lambda|$



$\text{Arg}(\lambda)$ gives rate of oscillation.

$$\vec{X}(n+1) = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \vec{X}(n) \qquad S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$\lambda = 2 \pm i$$

$$\vec{X}(0) = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \qquad S^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

$$\vec{y}(0) = S^{-1} \vec{X}(0) = \frac{1}{2i} \begin{pmatrix} 5 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \frac{5}{2i} \begin{pmatrix} i \\ 1 \end{pmatrix} - \frac{5}{2i} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$y_1(n+1) = (2+i)\, y_1(n) \qquad y_1(n) = (2+i)^n y_1(0)$$

$$y_2(n+1) = (2-i)\, y_2(n) \qquad y_2(n) = (2-i)^n y_2(0)$$

$$\cancel{\vec{y}(n)=}\ \vec{X}(n) = y_1(0)(2+i)^n \begin{pmatrix} i \\ 1 \end{pmatrix} + y_2(0)(2-i)^n \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$= \frac{5}{2i} (2+i)^n \begin{pmatrix} i \\ 1 \end{pmatrix} - \frac{5}{2i} (2-i)^n \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\frac{dx_1}{dt} = 2x_1 - x_2 \qquad \frac{dy_1}{dt} = (2+i)\, y_1$$

$$\frac{dx_2}{dt} = x_1 + 2x_2 \qquad \frac{dy_2}{dt} = (2-i)\, y_2$$

$$y_1(t) = e^{(2+i)t}\, y_1(0)$$

$$y_2(t) = e^{(2-i)t}\, y_2(0)$$

Def: $e^{(a+bi)t}$ = soln to $y' = (a+bi)y$ with $y(0) = 1$.

$$= e^{at}(\cos bt + i \sin bt)$$

$\mathrm{Re}\,\lambda$ gives growth rate

$\mathrm{Im}\,\lambda$ gives oscillation (angular) frequency

$$e^{zt} = \sum_{n=0}^{\infty} \frac{z^n t^n}{n!} \qquad \text{solves} \quad \frac{de^{zt}}{dt} = z\, e^{zt}$$

$$e^{At} = \sum \frac{A^n t^n}{n!} \qquad \text{solves} \quad \frac{d}{dt} e^{At} = A e^{At}$$

$$= S \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_m t} \end{pmatrix} S^{-1}$$

$$\vec{X}(t) = e^{At} \vec{X}(0)$$