$V =$ n-dimensional vector

basis $= \beta = \{\vec{b}_1, \cdots \vec{b}_n\}$

If $\vec{x} = c_1\vec{b}_1 + \cdots\ c_n\vec{b}_n$

Coordinates of $\vec{x}$ in β basis $= [x]_\beta = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

Ex1: $V = \{\text{Cubic polynomials in } t\}$

$\beta\ \cancel{\beta} = \{1, t, t^2, t^3\}$

$\vec{x} = 3 + 7t + 14t^2 - 45t^3 = 3\vec{b}_1 + 7\vec{b}_2 + 14\vec{b}_3 - 45\vec{b}_4$

$[\vec{x}]_\beta = \begin{pmatrix} 3 \\ 7 \\ 14 \\ -45 \end{pmatrix}$

Ex2: $V = \{2\times2 \text{ matrices}\}$.

$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

$\vec{x} = \begin{pmatrix} 1 & 3 \\ 5 & -10 \end{pmatrix} = \vec{b}_1 + 3\vec{b}_2 + 5\vec{b}_3 - 10\vec{b}_4$

$[x]_\beta = \begin{pmatrix} 1 \\ 3 \\ 5 \\ -10 \end{pmatrix}$

Ex3

$$V = \text{Col}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \beta = \left\{ \underset{b_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{b_2}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} \right\}$$

$$\vec{x} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 2\vec{b}_1 + \vec{b}_2$$

$$[\vec{x}] = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Ex4: $V = \left\{ \text{soln to } \frac{d^2 f(t)}{dt^2} = -4 f(t) \right\}$

$$\beta = \{ \cos(2t), \sin(2t) \}$$

$$\vec{x}(t) = 3\cos(2t) + 2\sin(2t) \qquad [x]_\beta = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Let $\beta = \{\vec{b}_1, \ldots \vec{b}_n\}$ is a basis for V

$\mathcal{D} = \{\vec{d}_1, \ldots \vec{d}_n\}$ is another basis.

$$S_{BD} = \begin{pmatrix} [d_1]_\beta & \cdots & [d_n]_\beta \end{pmatrix}$$

$$S_{DB} = \begin{pmatrix} [b_1]_D & \cdots & [b_n]_D \end{pmatrix}$$

<u>Thm</u> For any $\vec{x} \in V$,

$$[x]_B = S_{BD} [x]_D$$

$$[x]_D = S_{DB} [x]_B$$

$$S_{BD} = S_{DB}^{-1}$$

$$\vec{x} = c_1 \vec{d}_1 + \cdots \quad c_n \vec{d}_n \qquad [x]_D = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$[\vec{x}]_B = c_1 [d_1]_B + \cdots + c_n [\vec{d}_n]_B$$

$$= \begin{pmatrix} [d_1]_B & \cdots & [d_n]_B \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = S_{BD} \underbrace{[x]_D}$$

S_{BD} converts from D to B. (read right-to-left)

Ex: $V = \mathbb{R}^3$ $\quad \mathcal{D} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

$$\mathcal{B} = \left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\8\end{pmatrix} \right\}.$$

$\vec{x} = \begin{pmatrix}7\\2\\4\end{pmatrix}.$

$$b_1 = d_1 + 2d_2 + 3d_3$$

$$S_{DB} = \begin{pmatrix} [b_1]_D & [b_2]_D & [b_3]_D \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 8 \end{pmatrix}$$

$$S_{BD} = S_{DB}^{-1} = \begin{pmatrix} -8/3 & 10/3 & -1 \\ 8/3 & -13/3 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

$$[\vec{x}]_D = \begin{pmatrix}7\\2\\4\end{pmatrix}$$

$$[x]_B = \begin{pmatrix} -8/3 & 10/3 & -1 \\ 8/3 & -13/3 & 2 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix}7\\2\\4\end{pmatrix} = \begin{pmatrix}-16\\18\\-7\end{pmatrix}$$

$$\begin{pmatrix}7\\2\\4\end{pmatrix} = -16\begin{pmatrix}1\\2\\3\end{pmatrix} + 18\begin{pmatrix}4\\5\\6\end{pmatrix} - 7\begin{pmatrix}7\\8\\8\end{pmatrix}$$

$V =$ quadratic poly's in t

$\mathcal{D} = \{1, t, t^2\}$, $\cancel{\mathcal{B} = \{1+t+t^2,}$

$\mathcal{B} = \{1 + 2t + 3t^2, 4 + 5t + 6t^2, 7 + 8t + 8t^2\}$.

$\vec{x} = 7 + 2t + 4t^2$.

Problem: Find S_{BD}, S_{DB}, $[x]_D$, $[x]_B$

$$[b_1]_D = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad [b_2]_D = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \quad [b_3]_D = \begin{pmatrix} 7 \\ 8 \\ 8 \end{pmatrix}$$

$$S_{DB} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 8 \end{pmatrix} \qquad S_{BD} = S_{DB}^{-1}$$

$$[x]_D = \begin{pmatrix} 7 \\ 2 \\ 4 \end{pmatrix} \qquad [x]_B = S_{BD}\,[x]_D = \begin{pmatrix} -16 \\ 18 \\ -7 \end{pmatrix}$$

A <u>linear transformation</u> is a map

$L: V \to W$ s.t., for all $\vec{x}, \vec{y} \in V$
and all scalars c,

$$L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$$

$$L(c\vec{x}) = cL(\vec{x})$$

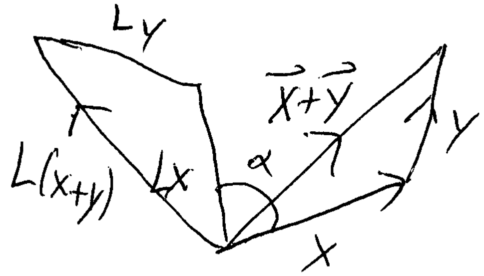
<u>Ex 1</u> $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, A is $m \times n$ matrix.

$L(\vec{x}) = A\vec{x}$.

$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ ✓

$A(c\vec{x}) = cA\vec{x}$ ✓

Ex 2. $V = W = \mathbb{R}^2$, L = rotation by angle α



Ex3 $V=\mathbb{R}^3$, $W=\mathbb{R}^2$, $L\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$

Projection.

Ex4 $V=$ any n-dim'l vector space, w/ basis $\mathcal{B}$.
$W=\mathbb{R}^n$.

$L(\vec{x})=[\vec{x}]_{\mathcal{B}}$

$[\vec{x}+\vec{y}]_{\mathcal{B}}=[\vec{x}]_{\mathcal{B}}+[\vec{y}]_{\mathcal{B}}$

$[c\vec{x}]_{\mathcal{B}}=c[\vec{x}]_{\mathcal{B}}$

Ex5 $V=$ cubic polys, $W=$ quadratic polys.
$L=\frac{d}{dt}$.

Ex5a $V=W=$ soln to $d^2f/dt^2=-4f$,
$L=d/dt$.

Ex6: $V=W=\{2\times 2 \text{ matrices}\}$

$L(x)=x^T$

Things that aren't linear.

$V=W=\mathbb{R}, \quad L(x)=|x|.$

$V=W=\mathbb{R}, \quad L(x)=x+1$

$V=W=\mathbb{R}, \quad L(x)=x^2.$

$V=W=$ functions on $[0,1]$.

$L(f(x))=|f(x)|$ or $L(f(x))=f''(x)+f^2.$

Back to linear......

Thm If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, ~~then there~~ and $L: V \to W$ is a linear transformation, then there is an $m \times n$ matrix A. s.t.

$L(x) = Ax$.

Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$, ... $\vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$\mathcal{E} = \{\vec{e}_1, \dots \vec{e}_n\} =$ "standard basis" for $\mathbb{R}^n$.

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{e}_1 + \cdots \; x_n \vec{e}_n$$

$$L(\vec{x}) = L(x_1 \vec{e}_1 + \cdots x_n \vec{e}_n)$$

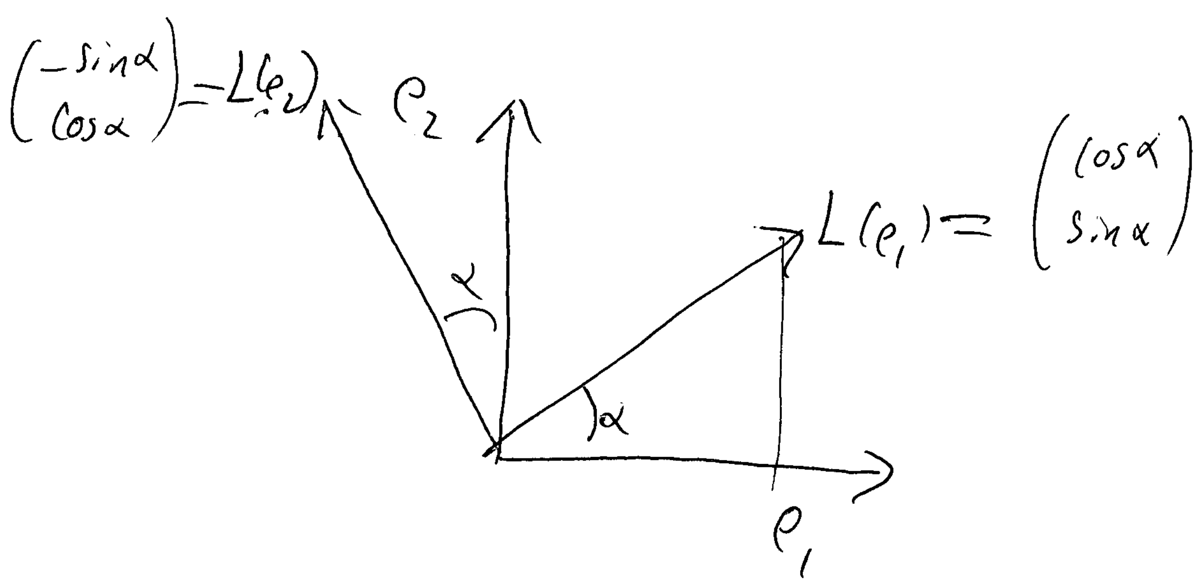
$$= L(x_1 \vec{e}_1) + \cdots + L(x_n \vec{e}_n) = x_1 L(\vec{e}_1) + \cdots x_n L(\vec{e}_n)$$

$$= \underbrace{\begin{pmatrix} L(\vec{e}_1) & \cdots & L(\vec{e}_n) \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = Ax$$

The matrix of the linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$ is

$$A = \begin{pmatrix} L(\vec{e}_1) & \cdots & L(\vec{e}_n) \\ \vdots & & \vdots \end{pmatrix}$$

$V = W = \mathbb{R}^2$, $L =$ Counter-clockwise rotation by α.



$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \text{rotation by } \alpha.$$

$$\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \text{rotation by } \beta.$$

Rotate by α and then rotate by β.

$$\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \vec{X}$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} \vec{X}$$

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{pmatrix}$$

$$\left(\text{Matrix of } L_1 \circ L_2\right) = \left(\begin{matrix}\text{Matrix} \\ \text{of } L_1\end{matrix}\right)\left(\begin{matrix}\text{Matrix} \\ \text{of } L_2\end{matrix}\right)$$

$V =$ n-diml vector space w/ basis $\mathcal{B}$

$W =$ m-diml vector space w/ basis $\mathcal{D}$

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, leftrightarrow] & W \arrow[d, leftrightarrow] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

$x \in V$

$$\begin{tikzcd} \vec{x} \arrow[r, "L"] \arrow[d, leftrightarrow] & L(\vec{x}) \arrow[d, leftrightarrow] \\ {[x]_B} \arrow[r, "A"] & {[L(x)]_D} \end{tikzcd}$$

Goal: Find A s.t.

$$[L(x)]_D = A\,[x]_B.$$

$$A = \begin{pmatrix} [L(b_1)]_D & \cdots & [L(b_n)]_D \end{pmatrix}$$

$= \text{matrix of } L \text{ w.r.t. bases } \mathcal{B} \text{ and } \mathcal{D}.$

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n \qquad [\vec{x}]_B = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$L(\vec{x}) = c_1 L(\vec{b}_1) + \cdots + c_n L(\vec{b}_n)$$

$$[L(\vec{x})]_D = c_1 [L(\vec{b}_1)]_D + \cdots + c_n [L(\vec{b}_n)]_D$$

$$= A [\vec{x}]_B$$

Ex: $V =$ cubic polys, $\mathcal{B} = \{1, t, t^2, t^3\}$

$W =$ quadratic polys, $\mathcal{D} = \{1, t, t^2\}$.

$L = \frac{d}{dt}$.

$L(b_1) = 0 \qquad [L(b_1)]_D = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$L(b_2) = 1 \qquad [L(b_2)]_D = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$L(b_3) = 2t \qquad [L(b_3)]_D = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$L(b_4) = 3t^2 \qquad [L(b_4)]_D = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

Matrix of L is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = A$.

$\vec{X} = a + bt + ct^2 + dt^3$

$L(\vec{x}) = b + 2ct + 3dt^2$

$$[L(\vec{x})]_D = \begin{pmatrix} b \\ 2c \\ 3d \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = A\,[x]_\mathcal{B}$$

Ex: $V = W =$ soln to $\frac{d^2f}{dt^2} = -4f$.

$\beta = \mathcal{D} = \{\cos(2t), \sin(2t)\}$.

$L = \frac{d}{dt}$.

$L(b_1) = -2\sin(2t)$

$[L(b_1)]_{\mathcal{D}} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$L(b_2) = 2\cos(2t)$ $\quad [L(b_2)]_{\mathcal{D}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

matrix of L is $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

A linear trans from a space to itself is called an <u>operator</u>.

$[L^2] = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$