$V =$ n dimensional vector space

$\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_n\} =$ basis for V.

$\mathcal{D} =$ another basis

$$\begin{tikzcd}
V \arrow[d, "{[x]_B}"] \arrow[dr, "{[x]_D}"] & \\
\mathbb{R}^n \arrow[r, shift left, "P_{DB}"] & \mathbb{R}^n \arrow[l, shift left, "P_{BD}"]
\end{tikzcd}$$

$$P_{DB}[x]_\mathcal{B} = [x]_D$$

$$P_{BD}[x]_D = [x]_\mathcal{B}$$

$$P_{BD} = \begin{pmatrix} [d_1]_B & \cdots & [d_n]_B \end{pmatrix}$$

$$P_{DB} = \begin{pmatrix} [b_1]_D & \cdots & [b_n]_D \end{pmatrix}$$

$$P_{DB} = P_{BD}^{-1}$$

V - basis $\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_n\}$

W — basis $\mathcal{D} = \{\vec{d}_1, \dots \vec{d}_m\}$.

$L : V \longrightarrow W$ linear transformation.

$$\begin{tikzcd}
x & V \arrow[r, "L"] \arrow[d, "{[\;]_B}"'] & W \arrow[d, "{[\;]_D}"] & L(x) \\
{[x]_B} & \mathbb{R}^n \arrow[r, "{[L]_{DB}}"] & \mathbb{R}^m & {[L(x)]_D \text{ or } [L]_{DB}[x]_B}
\end{tikzcd}$$

$[L]_{DB} =$ matrix of L w.r.t. bases $\mathcal{B}, \mathcal{D}$.

$$[L]_{DB}[x]_B = [L(x)]_D$$

$$[L]_{DB} = \left([L(\vec{b}_1)]_D \cdots [L(b_n)]_D \right)$$

$$\begin{tikzcd}
\text{Cubic poly's} = V \arrow[r, "\frac{d}{dt}"] \arrow[d, "B=\{1,t,t^2,t^3\}"'] & W = \text{quadratic poly's} \arrow[d, "\mathcal{D}=\{1,t,t^2\}"] \\
\mathbb{R}^4 \arrow[r, "\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}"] & \mathbb{R}^3
\end{tikzcd}$$

If $V=W$, $L=$ identity

$$\begin{tikzcd}
x \quad V \arrow[r, "\text{identity}"] \arrow[d, "[\;]_B"'] & V \quad x \arrow[d, "[x]_D"] \\
[x]_B \quad \mathbb{R}^n \arrow[r, "[id]_{DB} = P_{DB}"] & \mathbb{R}^n \quad [x]_D
\end{tikzcd}$$

$L : V \longrightarrow W$

Kernel of $L = \ker(L) = \{x \mid L(x) = 0\}$

$\subset V$

Image $(L) = \{L(x)\}. \subset W.$

(Range (L)) $=$ all possible outputs.

Claim: $x \in \ker(L) \longleftrightarrow [x]_B \in \mathrm{Nul}\,[L]_{DB}$.

$$L(x) = 0$$

$$\Updownarrow$$

$$[L(x)]_D = 0$$

$$\Updownarrow$$

$$[L]_{DB}[x]_B = 0$$

$$\Updownarrow$$

$$[x]_B \in \mathrm{Nul}\left([L]_{DB}\right).$$

<u>Claim:</u> $\cancel{\Phi}$ $y \in Im(L) \iff [y]_D \in Col([L]_{DB})$

$$y \in Im(L)$$
$$\Updownarrow$$
$$y = L(x) \quad \text{for some } x \in V.$$
$$\Updownarrow$$
$$[y]_D = [L(x)]_D \qquad ''$$
$$\Updownarrow$$
$$[y]_D = [L]_{DB}[x]_B \quad \text{for some } x.$$
$$\Updownarrow$$
$$[y]_D \in Col([L]_{DB})$$

Dim Col(A) + Dim Nul(A) is n (if A is $n \times m$).

pivots

free variables.

$$\boxed{\begin{array}{l} \text{Dim } Im(L) + \text{Dim } Ker(L) = \text{Dim}(V) \\ \text{Dimension theorem.} \end{array}}$$

$V=$ cubic polys $\quad W=$ quadratic $\quad L=\frac{d}{dt}$

basis $\{1,t,t^2,t^3\}$ $\quad$ basis $=\{1,t,t^2\}$

$$[L]_{DB}=\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix} \qquad \text{Rank } [L]_{DB}=3$$

Col $[L]_{DB}$ is 3-dim'l,

so $\operatorname{Im} L$ is 3 dim'l.

Since $\operatorname{Im} L \subset W$ and $\dim(W)=3$,

$\operatorname{Im} L = W$.

Basis for Nul $[L]_{DB}=\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}$

Basis for Ker $L=\{1+0t+0t^2+0t^3\}$

$=$ constant functions.

$$W \xrightarrow{\int dt} V \qquad \int 1\,dt = t$$

$$\int t\,dt = t^2/2$$

$$\int t^2 dt = t^3/3$$

$$[S]_{BD}=\begin{pmatrix}0&0&0\\1&0&0\\0&1/2&0\\0&0&1/3\end{pmatrix}$$

$$[L]_{DB}\,[S]_{BD} = [L \circ S]_{DD} = [id]_{DD} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= [S \circ L]_{BB}$$

Specialize to $W = V$

Output basis = input basis

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "[\ ]_B"'] & V \arrow[d, "[\ ]_B"] \\ \mathbb{R}^n \arrow[r, "[L]_{BB}"] & \mathbb{R}^n \end{tikzcd}$$

L is an "operator".

~~Claim~~ Def: L^{-1} is an operator s.t. $L \circ L^{-1} = L^{-1} \circ L$ = identity.

Def x is an eigenvector of L w/ e-val λ if $x \neq 0$ and $L(x) = \cancel{x} \lambda x$.

Ex: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is e-vec of (transpose) w/ e-val -1.

e^{2t} is e-vec of $\frac{d}{dt}$ w/ e-val 2.

$\sin(t)$ is e-vec of $\frac{d^2}{dt^2}$ w/ e-val -1

Prop:

$$[L^n]_{BB} = \left([L]_{BB}\right)^n$$

$$[L^{-1}]_{BB} = \left([L]_{BB}\right)^{-1}$$

X is e-vec of L w/ e-val λ

$$\Updownarrow$$

$[X]_B$ is e-vec of $[L]_{BB}$ w/ e-val λ.

---

pf of last bit.

$$L(x) = \cancel{\lambda x}\ \lambda x$$

$$[L(x)]_B = \lambda [x]_B$$

$$[L]_{BB} [x]_B = \lambda [X]_B$$

Change-of-basis formula

Thm If B, D are bases for V and $L: V \to V$ is an operator,

$$[L]_{DD} = P_{DB} [L]_{BB} P_{BD}$$

$$= P_{DB} [L]_{BB} P_{DB}^{-1}$$

$$= P_{BD}^{-1} [L]_{BB} P_{BD}$$

Compare to

$$[x]_D = P_{DB} [x]_B$$

Pf: Let $x \in V$.

$$[L(x)]_D \overset{?}{=} P_{DB} [L]_{BB} P_{BD} [x]_D$$

$$= P_{DB} [L]_{BB} [x]_B$$

$$= P_{DB} [L(x)]_B$$

$$= [L(x)]_D \quad \checkmark$$

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "{[L]_{DD}}"] \arrow[dd, shift left, "P_{BD}"] & & & \mathbb{R}^n \arrow[dd, shift right, "P_{BD}"'] \\
& V \arrow[ul, "{[\ ]_D}"'] \arrow[dl, "{[\ ]_B}"] \arrow[r, "L"] & V \arrow[ur, "{[\ ]_D}"] \arrow[dr, "{[\ ]_B}"] & \\
\mathbb{R}^n \arrow[rrr, "{[L]_{BB}}"] \arrow[uu, shift left, "P_{\cancel{BD}\,DB}"] & & & \mathbb{R}^n \arrow[uu, shift right, "P_{\cancel{BD}\,DB}"']
\end{tikzcd}
$$

Suppose $\{\vec{b}_1, \dots \vec{b}_n\}$ is a basis of e-vec's of L.

$$[L]_{BB} = \begin{pmatrix} [L(b_1)]_B & \cdots\cdots & [L(b_n)]_B \end{pmatrix}$$

$$= \begin{pmatrix} [\lambda_1 b_1]_B & \cdots\cdots & [\lambda_n \vec{b}_n]_B \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & & 0 \\ 0 & \vdots & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & \lambda_n \end{pmatrix}$$

Bases of e-vecs make operators look like diagonal matrices.

$V = W = \mathbb{R}^n$.

$D =$ Standard basis

$\beta =$ basis of e-vec.

$$P_{DB} = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} = S$$

$L(x) = Ax$.

$$[L]_{DD} = A.$$

$$[L]_{BB} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \Lambda$$

$$[L]_{DD} = P_{DB} [L]_{BB} P_{BD} = S \Lambda S^{-1}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\vec{y}(0) \arrow[r, "{\Lambda^n,\ e^{\Lambda t}}"] & \vec{y}(\text{later}) \arrow[d, "P_{DB}=S"] \\
\vec{x}(0) \arrow[u, "P_{BD}=S^{-1}"] \arrow[r, "{A^n,\ e^{At}}"] & \vec{x}(\text{later})
\end{tikzcd}$$

$$\vec{y} = [x]_\beta$$