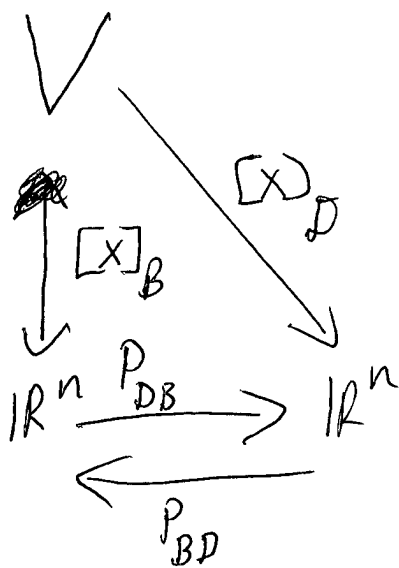


$V = n$  dimensional vector space

$\beta = \{\vec{b}_1, \dots, \vec{b}_n\} =$  basis for  $V$ .

$\mathcal{D} =$  another basis



$$P_{DB} [x]_B = [x]_D$$

$$P_{BD} [x]_D = [x]_B$$

$$P_{BD} = \begin{pmatrix} [d_1]_B & \dots & [d_n]_B \end{pmatrix}$$

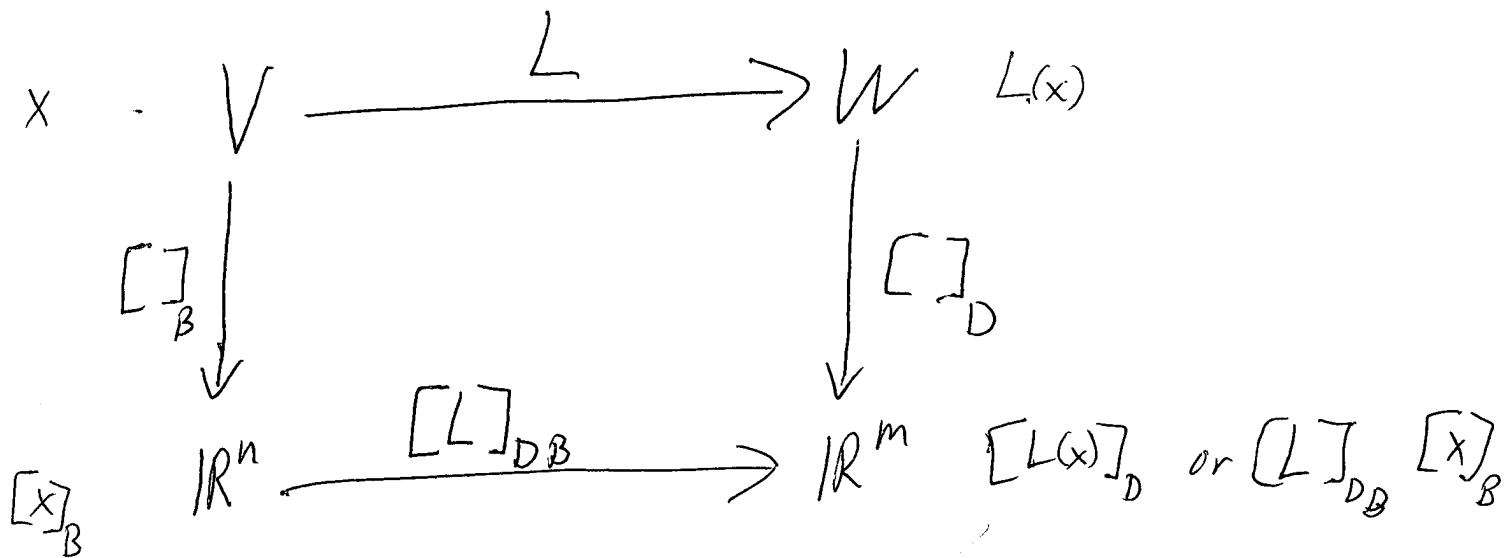
$$P_{DB} = \begin{pmatrix} [b_1]_D & \dots & [b_n]_D \end{pmatrix}$$

$$P_{DB} = P_{BD}^{-1}$$

$V$  - basis  $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$

$W$  - basis  $\mathcal{D} = \{\vec{d}_1, \dots, \vec{d}_m\}$ .

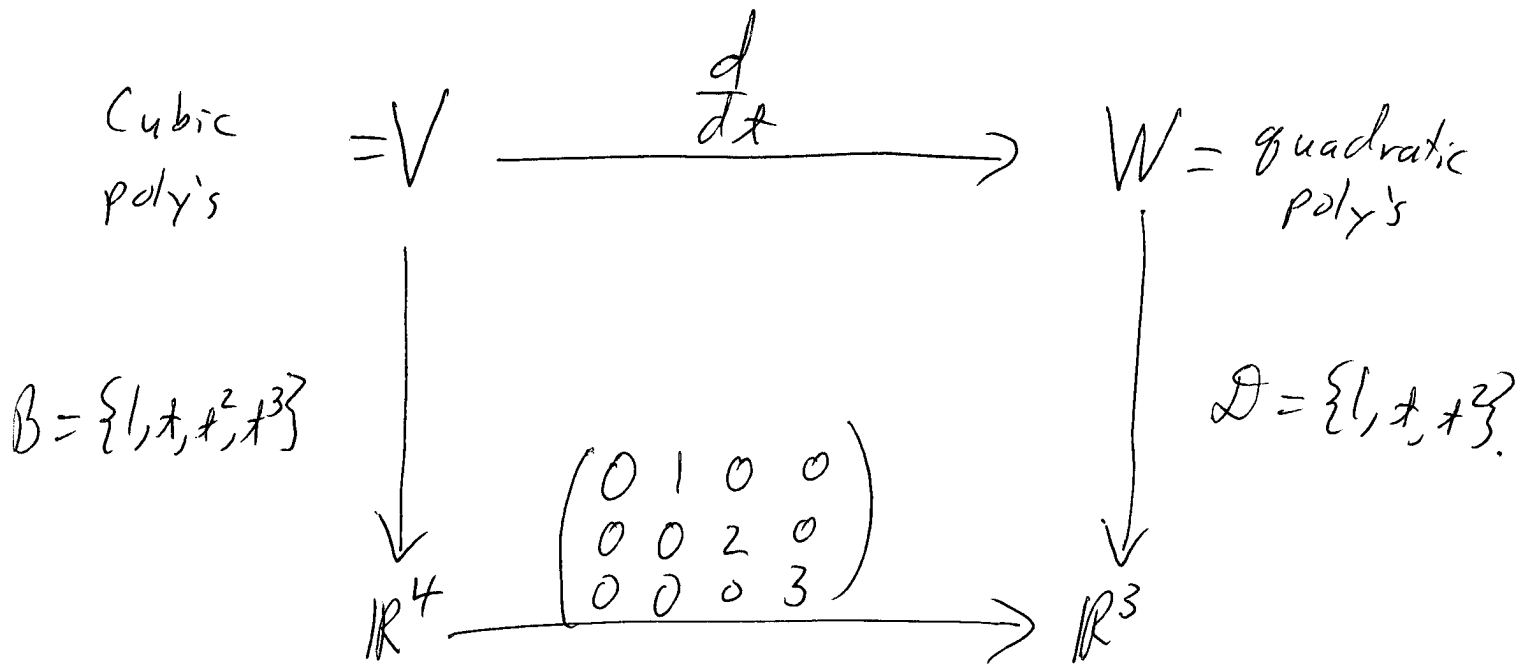
$L: V \rightarrow W$  linear transformation.



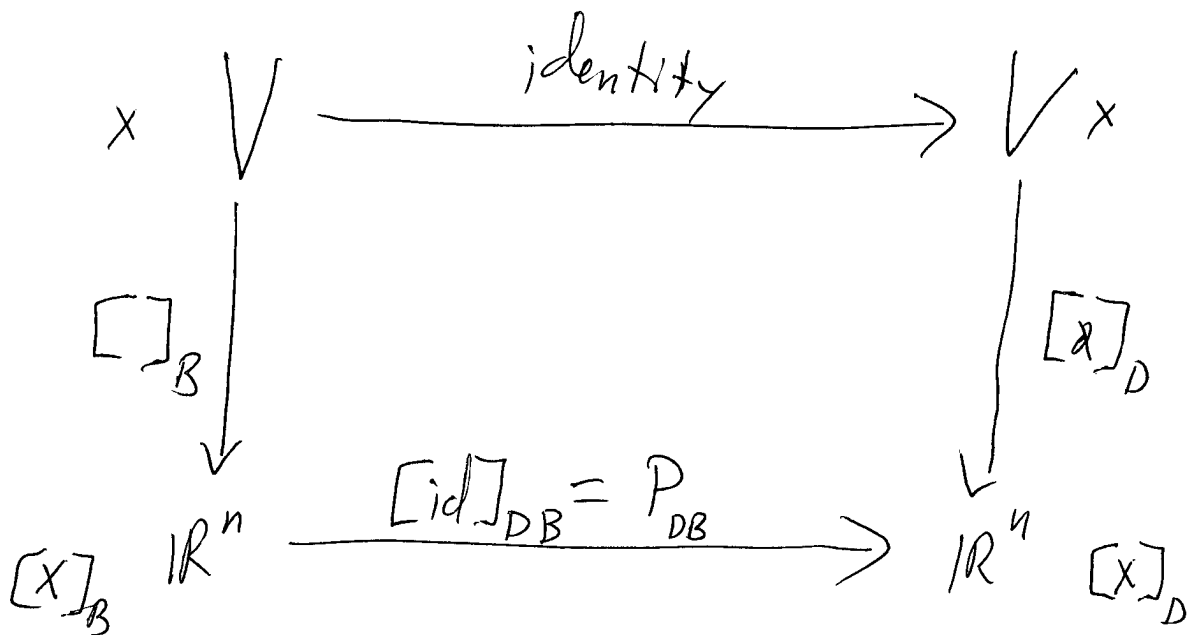
$[L]_{DB}$  = matrix of  $L$  wrt. bases  $\mathcal{B}, \mathcal{D}$ .

$$[L]_{DB} [x]_B = [L(x)]_D$$

$$[L]_{DB} = \left( [L(\vec{b}_1)]_D \quad \dots \quad [L(\vec{b}_n)]_D \right)$$



If  $V=W$ ,  $L = \text{identity}$



$$L: V \longrightarrow W$$

$$\text{Kernel of } L = \ker(L) = \{x \mid L(x) = 0\} \\ \subset V$$

$$\text{Image}(L) = \{L(x)\} \subset W.$$

(Range(L)) = all possible outputs.

$$\text{Claim: } x \in \ker(L) \iff [x]_B \in \text{Nul}([L]_{DB})$$

$$L(x) = 0$$

$$\iff$$

$$[L(x)]_D = 0$$

$$\iff$$

$$[L]_{DB} [x]_B = 0$$

$$\iff$$

$$[x]_B \in \text{Nul}([L]_{DB}).$$

Claim:  $y \in \text{Im}(L) \iff [y]_D \in \text{Col}([L]_{DB})$

$$y \in \text{Im}(L)$$

$$\iff$$

$$y = L(x) \text{ for some } x \in V.$$

$$\iff$$

$$[y]_D = [L(x)]_D \quad "$$

$$\iff$$

$$[y]_D = [L]_{DB} [x]_B \text{ for some } x.$$

$$\iff$$

$$[y]_D \in \text{Col}([L]_{DB})$$

$\text{Dim Col}(A) + \text{Dim Nul}(A)$  is  $n$   
 (if  $A$  is  $n \times m$ ).  
 $\uparrow$   $\uparrow$   
 pivots free variables.

$$\text{Dim Im}(L) + \text{Dim Ker}(L) = \text{Dim}(V)$$

Dimension theorem.

$V =$  cubic polys  
basis  $\{1, t, t^2, t^3\}$

$W =$  quadratic  
basis  $= \{1, t, t^2\}$

$$L = \frac{d}{dt}$$

$$[L]_{DB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\text{Rank } [L]_{DB} = 3$$

$\text{Col } [L]_{DB}$  is 3-diml,

so  $\text{Im } L$  is 3-diml.

Since  $\text{Im } L \subset W$  and  $\dim(W) = 3$ ,

$$\text{Im } L = W.$$

Basis for  $\text{Nul } [L]_{DB} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

Basis for  $\text{Ker } L = \{1 + 0t + 0t^2 + 0t^3\}$

= constant functions.

$$W \xrightarrow{\int dt} V$$

$$[S]_{BD} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

$$\int 1 dt = t$$

$$\int t dt = t^2/2$$

$$\int t^2 dt = t^3/3$$

$$[L]_{DB} [S]_{BD} = [L \circ S]_{DD} = [id]_{DD} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

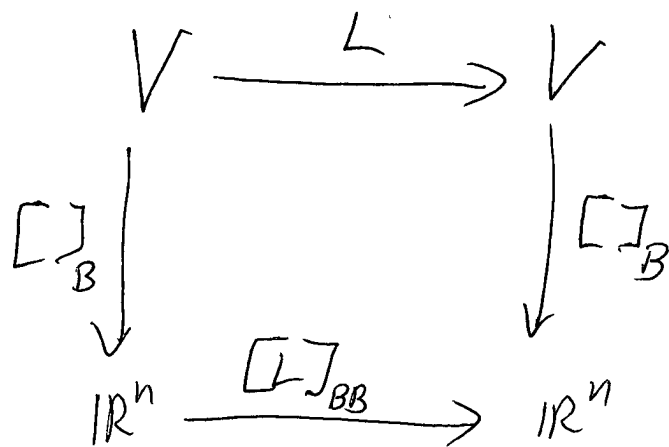
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$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= [S \circ L]_{BB}$$

Specialize to  $W=V$

Output basis = input basis



$L$  is an  
"operator".

---

~~Claim~~ Def:  $L^{-1}$  is an operator st.  $L \circ L^{-1} = L^{-1} \circ L = \text{identity}$ .

Def  $x$  is an eigenvector of  $L$  w/ e-val  $\lambda$   
if  $x \neq 0$  and  $L(x) = \lambda x$ .

Ex:  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  is e-vec of (transpose) w/ e-val  $-1$ .

$e^{2t}$  is e-vec of  $\frac{d}{dt}$  w/ e-val  $2$ .

$\sin(t)$  is e-vec of  $\frac{d^2}{dt^2}$  w/ e-val  $-1$ .



Prop.

$$[L^n]_{BB} = \left( [L]_{BB} \right)^n$$

$$[L^{-1}]_{BB} = \left( [L]_{BB} \right)^{-1}$$

$x$  is e-vec<sub>1</sub> of  $L$  w/ e-val  $\lambda$



$[x]_B$  is e-vec of  $[L]_{BB}$  w/ e-val  $\lambda$ .

---

pf of last bit.

$$L(x) = \lambda x$$

$$[L(x)]_B = \lambda [x]_B$$

$$[L]_{BB} [x]_B = \lambda [x]_B$$

# Change-of-basis formula

Thm If  $B, D$  are bases for  $V$  and  $L: V \rightarrow V$  is an operator,

$$\begin{aligned} [L]_{DD} &= P_{DB} [L]_{BB} P_{BD} \\ &= P_{DB} [L]_{BB} P_{DB}^{-1} \\ &= P_{BD}^{-1} [L]_{BB} P_{BD} \end{aligned}$$

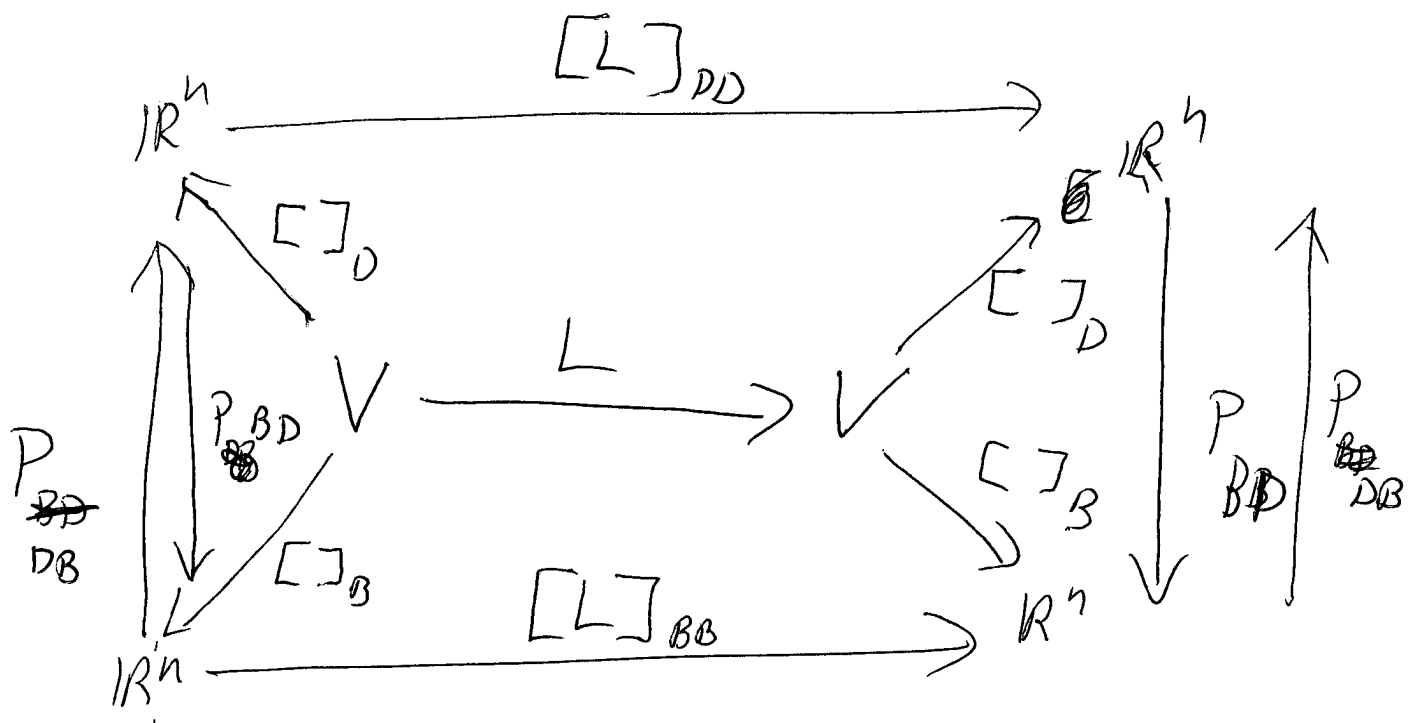
Compare to

$$[x]_D = P_{DB} [x]_B$$

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Pf: Let  $x \in V$ .

$$\begin{aligned} [L(x)]_D &\stackrel{?}{=} P_{DB} [L]_{BB} P_{BD} [x]_D \\ &= P_{DB} [L]_{BB} [x]_B \\ &= P_{DB} [L(x)]_B \\ &= [L(x)]_D \quad \checkmark \end{aligned}$$



Suppose  $\{\vec{b}_1, \dots, \vec{b}_n\}$  is a basis of  
e-vecs of  $L$ .

$$[L]_{BB} = \begin{pmatrix} [L(\vec{b}_1)]_B & \dots & [L(\vec{b}_n)]_B \end{pmatrix}$$

$$= \begin{pmatrix} [\lambda_1 \vec{b}_1]_B & \dots & [\lambda_n \vec{b}_n]_B \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & & & 0 \\ 0 & \vdots & & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & & \lambda_n \end{pmatrix}$$

Bases of e-vecs make operators  
look like diagonal matrices.

$$V=W=\mathbb{R}^n.$$

$D =$  Standard basis

$\beta =$  basis of e-vec.

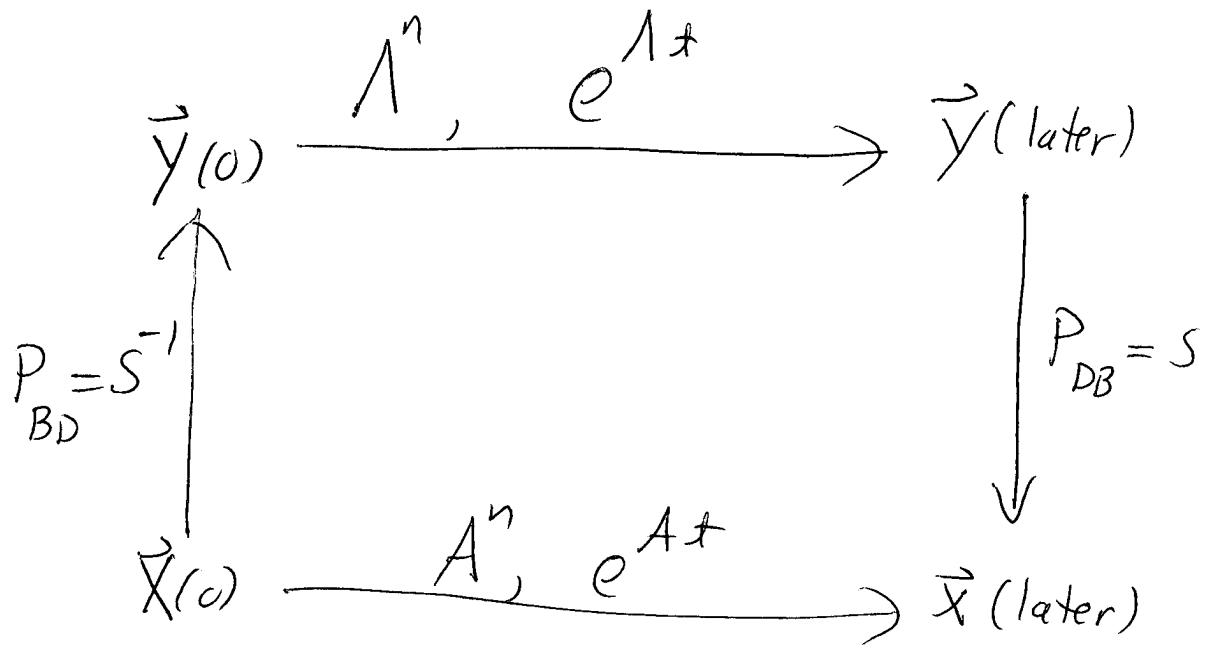
$$P_{DB} = (b_1, \dots, b_n) = S$$

$$L(x) = Ax.$$

$$[L]_{DD} = A.$$

$$[L]_{\beta\beta} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \Lambda$$

$$[L]_{DD} = P_{DB} [L]_{\beta\beta} P_{\beta D} = S \Lambda S^{-1}$$



$$\vec{y} = [x]_B$$