What is a matrix?

1) An array $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ of #s.

2) A collection $\begin{pmatrix} \vec{a_1} & \cdots & \vec{a_n} \end{pmatrix}$ of vectors.

3) A collection $\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}$ of m linear expressions ~~equations~~ in $\mathbb{R}^n$

4) A linear transformation $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

What does $A\vec{x}$ mean?

1) $(Ax)_i = \sum_j a_{ij} x_j$

2) $A\vec{x} = x_1 \vec{a_1} + \cdots + x_n \vec{a_n}$ = linear combination of columns of A.

3) $A\vec{x} = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}$

4) $L(\vec{x})$

What does AB mean?

1) $(AB)_{ik} = \sum_j A_{ij} B_{jk}$

2) $(AB) = \begin{pmatrix} A(\vec{b}_1) & \cdots & A(\vec{b}_n) \end{pmatrix}$

3) Linear transformation: First do B, then A.

What does it mean to solve $A\vec{x} = \vec{b}$.

1) Write $\vec{b}$ as a lin comb of columns of A.

2) Solved m equations in n unknowns

3) Find all $\vec{x}$ that map to $\vec{b}$ (Linear trans).

How do you solve $A\vec{x}=\vec{b}$?

[Digression: If you know A^{-1}, $\vec{x}=A^{-1}\vec{b}$]

In general, row reduce $[A \mid b]$.

REF good. $(A=LU)$

RREF better.

Rows w/o pivots $\Rightarrow$ possible contradictions.

Columns w/o pivots $\Rightarrow$ free variables.

Rows w/ pivots $\Rightarrow$ Give pivot variables in terms of free variables

---

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 3 & a \\ 0 & 0 & 1 & -2 & b \\ 0 & 0 & 0 & 0 & c \end{array}\right)$$

$\uparrow$ x_1 free $\qquad$ $\uparrow$ x_4 free

If $c \neq 0$ contradiction.

If $c = 0$ keep going.

$$\begin{aligned} x_1 &= \quad x_1 \\ x_2 &= a \quad - 3x_4 \\ x_3 &= b \quad + 2x_4 \\ x_4 &= \quad x_4 \end{aligned}$$

$$\vec{x} = \begin{pmatrix} 0 \\ a \\ b \\ 0 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -3 \\ 2 \\ 1 \end{pmatrix}$$

particular solution

special soln, basis for Nul(A)

Rank (A) = # pivots $= k$.

If $k=m$, then

1) $Ax=b$ has a soln for every $\vec{b} \in \mathbb{R}^m$
2) Every vector in $\mathbb{R}^m$ is a lin comb of columns of A.
3) Span (columns) $= \mathbb{R}^m$
4) A is onto
5) A has a right-inverse. $AB = I_m$

If $k \neq m$, none of these are true

If $k=n$

1) $Ax=b$ has at most one soln.
2) $Ax=0$ has exactly one soln.
3) Columns of A are linearly independent.
4) A is 1-1
5) A has a left-inverse $CA = I_n$

Note: $k \leq \min(m,n)$

If $m<n$, $k<n$

If $m>n$, $k<m$

If $m=n$, either $k=m=n$, or $k\neq m$ and $k \neq n$

If $m=n=k$,

1) $A\vec{x}=\vec{b}$ has unique soln for each $\vec{b}$
2) A is 1-1
3) A is onto
4) A has an inverse: $AA^{-1}=A^{-1}A=I$
5) Columns are lin ind
6) Columns span $\mathbb{R}^n$
7) Columns form a basis for $\mathbb{R}^n$
8) $\operatorname{Det} A \neq 0$
9) A is "invertible"
10) $\operatorname{RREF}(A) = I$

Find A^{-1} by row-reducing $(A \mid I)$ to $(I \mid A^{-1})$

If $m=n$ but $k<n$, none of these apply and A is "singular".

A Vector space is a set where you cantake linear combinations.

(You can add vectors, you can multiply them by scalars, arithmetic works as usual).

(Lots of examples)

A subset W of a vector space V is a subspace if

1) W is closed under addition
2) W " " " scalar mult.
3) $0 \in W$

4 fundamental subspaces associated to an ~~ma~~ $m \times n$ matrix A.

$\mathrm{Col}(A) =$ Span {Columns of A}

$= \{A\vec{x}\}$ = Image of linear transformation.

$=$ Subspace of $\mathbb{R}^m$

good basis = pivot columns of A dim $= K$

$\neq$ pivot columns of A_{rref}

$\mathrm{Row}(A) = \mathrm{Col}(A^T) =$ span {rows of A}. dim $= K$

$=$ Subspace of $\mathbb{R}^n$

good basis = ~~row~~ {non-zero rows of A_{rref}}

Row operations don't change $\mathrm{Row}(A)$

they <u>do</u> change $\mathrm{Col}(A)$.

$\mathrm{Nul}(A) =$ Solutions to $A\vec{x} = 0$ dim $= n - K$

$=$ Subspace of $\mathbb{R}^n$

$= (\mathrm{Row}(A))^\perp$

good Basis = special solns $\begin{pmatrix} \text{one free} = 1 \\ \text{others} = 0 \end{pmatrix}$

$\mathrm{Coker}(A) = \mathrm{Nul}(A^T)$, basis = special soln to $A^T x = 0$.

dim $= m - K$

Orthogonality and transposes.

$$\vec{x} \cdot \vec{y} = x^T y$$

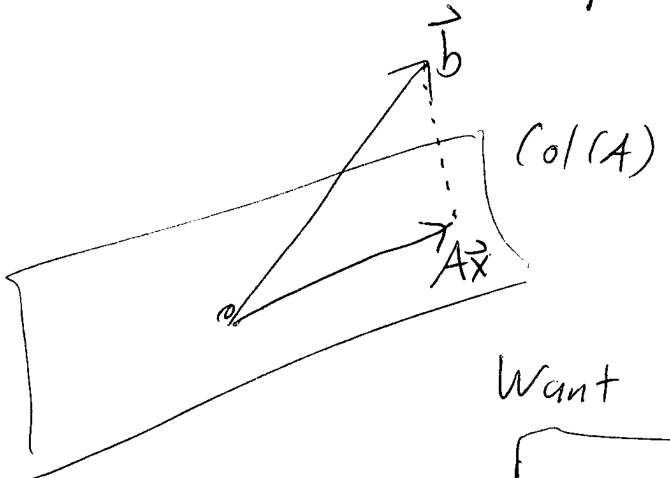
If $A = (\vec{a}_1 \cdots \vec{a}_n)$

$$A^T x = \begin{pmatrix} a_1 \cdot x \\ \vdots \\ a_n \cdot x \end{pmatrix}$$

$\mathrm{Nul}(A) = (\mathrm{Row}(A))^\perp$

$\mathrm{Nul}(A^T) = (\mathrm{Col}(A))^\perp$

Problem: Find x s.t. $|Ax - b|$ is minimized.



Want $Ax - b \perp \mathrm{Col}(A)$.

Want $A^T(Ax-b) = 0$

$$\boxed{A^T A x = A^T b}$$

A <u>least-squares</u> soln to $Ax=b$ is

a true soln to $A^TA\,x = A^Tb$.

Problem: Find pt in $Col(A)$ closest to $\vec{b}$.

Ans: $A\vec{x}$, where $A^TA\vec{x} = A^T\vec{b}$

$A\vec{x}$ = projection of $\vec{b}$ onto $Col.(A)$.

Orthogonal bases are great.

If $\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_n\}$ is an orthogonal basis,

and if $\vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$

$$c_j = \frac{\vec{x}\cdot\vec{b}_j}{\vec{b}_j\cdot\vec{b}_j} = \text{"scalar projection"}$$

$c_j\vec{b}_j$ = part of $\vec{x}$ in $\vec{b}_j$ direction

$$= \frac{\vec{x}\cdot\vec{b}_j}{\vec{b}_j\cdot\vec{b}_j}\vec{b}_j = P_{b_j}\vec{x} = \text{vector Projection.}$$

$$P_{b_j} = \frac{b_j \, b_j^T}{(b_j \cdot b_j)}$$

E.g., if $b_j = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $P_{b_j} = \frac{1}{14} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

Gram- Schmidt

Start with basis $\{\vec{X}_1, .. \vec{X}_n\}$

Convert to orthoganal basis $\{\vec{Y}_1, ... \vec{Y}_n\}$

Convert to orthonormal basis $\{\vec{Z}_1, ... \vec{Z}_n\}$.

$$\vec{Y}_1 = \vec{X}_1$$

$$\vec{Y}_2 = \vec{X}_2 - P_{\vec{Y}_1} \vec{X}_2$$

$$\vec{Y}_3 = \vec{X}_3 - P_{\vec{Y}_1} \vec{X}_3 - P_{\vec{Y}_2} \vec{X}_3$$

NOT $P_{X_2} \vec{X}_3$

etc.

$$\vec{Z}_K = \frac{\vec{Y}_K}{|\vec{Y}_K|}$$

If $S = \begin{pmatrix} \vec{y}_1 & \cdots & \vec{y}_n \end{pmatrix}$ $\quad \vec{y}_i \perp \vec{y}_j$

$$S^{-1} = \begin{pmatrix} \frac{1}{y_1 \cdot y_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{y_n \cdot y_n} \end{pmatrix} \begin{pmatrix} y_1^T \\ \vdots \\ y_n^T \end{pmatrix}$$

If y's orthonormal, $S^{-1} = S^T$ Orthogonal matrix.

A linear transformation is a map $\cancel{L: \mathbb{R}^n \longrightarrow \mathbb{R}^m}$

$$L: V \longrightarrow W$$

s.t.
$$L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$$
$$L(c\vec{x}) = cL(\vec{x})$$

All linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ are matrices

$$A = \begin{pmatrix} L(e_1) & \cdots & L(e_n) \end{pmatrix}$$

$$L: V \longrightarrow W$$

$$
\begin{tikzcd}
V \arrow[r, "L"] \arrow[d, "{[\ ]_\beta}"'] & W \arrow[d, "{[\ ]_D}"] \\
\mathbb{R}^n \arrow[r, "{[L]_{DB}}"] & \mathbb{R}^m
\end{tikzcd}
$$

$$[L]_{DB}[x]_\beta = [L(x)]_D$$

$$[L]_{DB} = \left([L(b_1)]_D \cdots \quad [L(b_n)]_D \right)$$