Big ideas of power series.

1) Radius and interval ~~co~~ of convergence.

$$\sum c_n (x-a)^n \begin{cases} \text{converges}_{\wedge}^{\text{absolutely}} & \text{if } |x-a|<R \\ \text{diverges} & \text{if } |x-a|>R \end{cases}$$

Find R with root or ratio test.

If $L = \lim \left|\frac{c_{n+1}}{c_n}\right|$, $R = \frac{1}{L}$

If $L = \lim |c_n|^{1/n}$, $R = 1/L$

2) Power series are functions. Manipulate like polynomials, <u>within radius of convergence.</u>

If $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$, then

$$f(x)+g(x) = \sum (a_n + b_n) x^n$$

$$f'(x) = \sum n a_n x^{n-1}$$

$$37 f(x) = \sum 37 a_n x^n$$

$$\int_0^x f(t)\,dt = \sum a_n \frac{x^{n+1}}{n+1}$$

$$x f(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

3) Functions are power series:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \text{ where}$$

$$c_n = \frac{f^{(n)}(a)}{n!} \qquad f^{n}(a) = n!\, c_n$$

4) Basic examples:

$$\frac{1}{1-x} = 1 + x + x^2 + \dots\dots = \sum_{n=0}^{\infty} x^n \qquad R = 1$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = \infty$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^k \quad \text{where} \quad (R=1)$$

$$\binom{k}{n} = \frac{k(k-1)\dots(k+1-n)}{n!}$$

5) Clever tricks.

If $f(x) = \sum a_n x^n$, then

$f(\text{Anything}) = \sum a_n (\text{Anything})^n$

e.g. $e^{5x^2} = \sum \frac{(5x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{5^n x^{2n}}{n!}$

~~ln(1+x)~~

$x^3 e^{5x^2} = \sum_{n=0}^{\infty} \frac{5^n x^{2n+3}}{n!}$

$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int 1 - x + x^2 - x^3 + \dots \,dx$ $\qquad R=1$

$= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$

$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx = \int 1 - x^2 + x^4 + x^6 + x^8 + \dots \,dx$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ $\qquad R=1$

6)

Remainders

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \text{first } n \text{ terms of Taylor Series}$$

<u>Thm</u> $f(x) = T_n(x) + \dfrac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$

for some # c between a and x

<u>Cor</u> If $f^{(k+1)}(x)$ is never bigger than M,

$$|f(x) - T_n(x)| < \frac{M}{(k+1)!}(x-a)^{k+1}$$

7) Uses:

a) Evaluating functions (approximately)

b) Evaluating limits

c) Approximating integrals

d) Solving differential eq.s