M408S Final Exam Solutions, May 13, 2013

1. Areas and volumes. (8 pts) Let R be the region in the x-y plane between the x axis, the curve $y = 1/x$ and the line $x = 1$. Let V be the 3-dimensional solid obtained by rotating R around the x axis.

a) Express the area of R as an improper integral. Does this integral converge? If so, evaluate the integral. If not, explain why it diverges.

The area is $\int_1^\infty \frac{dx}{x}$, which diverges since $\ln(t)$ goes to ∞ as $t \to \infty$.

b) Express the volume of V as an improper integral. Does this integral converge? If so, evaluate the integral. If not, explain why it diverges.

$V = \int_1^\infty \pi(1/x)^2 dx = \pi \int_1^\infty \frac{dx}{x^2}$ which converges to π.

2. Integrals and limits. (12 pts) Evaluate the following integrals and limits. If an improper integral or a limit does not exist, say "does not exist" or "diverges".

a) $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$.

Use the substitution $x = 2\sin(\theta)$ to convert the integral to $\int d\theta = \theta = \sin^{-1}(x/2)$. Evaluating at 1 and at 0 gives $\sin(1/2) - \sin(0) = \pi/6$.

b) $\lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x^3}$.

You can either apply L'Hospital's rule three times or use the Taylor expansion of $\sin(x)$ and $\cos(x)$:

$$\sin(x) - x\cos(x) = (x - x^3/6 + \cdots) - x(1 - x^2/2 + \cdots) = x^3/3 + \cdots.$$

Dividing by x^3 and taking a limit as $x \to 0$ gives $1/3$.

c) The sequence $\{a_n\}$ with $a_n = \frac{\sin(n)e^n}{ne^n + 1}$.

This sequence is sandwiched between $-e^n/(ne^n + 1)$ and $e^n/(ne^n + 1)$, both of which go to zero, so $a_n \to 0$.

3. (16 pts) For each of these series, indicate whether the series converges absolutely, converges conditionally, or diverges. Give a short explanation of why (e.g. "converges by comparison to $\sum 2^{-n}$").

a) $\sum_{n=1}^\infty \frac{\sin(\pi n/2)}{n}$.

This converges conditionally. The non-zero terms are $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

which is an alternating series (and adds up to $\pi/4$). Since $\sum \frac{1}{2n+1}$ diverges (by limit comparison to $\sum 1/n$), the original series does NOT converge absolutely, just conditionally.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n\sin(1/n)}{n^2+1}$.

This converges absolutely, by limit comparison to $\sum \frac{1}{n^2+1}$, since $n\sin(1/n) = \sin(1/n)/(1/n)$ approaches 1 as $n \to \infty$ (which you can see either with Taylor series or L'Hospital's rule).

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$.

This converges absolutely by the ratio test, with $R = 0$.

d) $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2(-1)^n + e^{-n}}{n^2+4n+1}$.

This diverges by the divergence test. For large n, a_n is either close to 5 or -5.

4. (10 points) Taylor polynomials

a) Compute the second order Taylor polynomial $T_2(x)$ for $f(x) = \tan(x)$ around $a = \pi/4$.

Since $f'(x) = \sec^2(x)$ and $f''(x) = 2\sec^2(x)\tan(x)$, we have $f(a) = 1$, $f'(a) = 2$ and $f''(a) = 4$, so $T_2(x) = 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2$.

b) Use this polynomial to approximate $\tan(\frac{\pi}{4} + 0.1)$.

$T_2(\frac{\pi}{4}+0.1) = 1+2(0.1)+2(0.1)^2 = 1.22$. The actual value of $\tan(\frac{\pi}{4}+0.1)$ is around 1.223049.

5. Partial derivatives. (8 pts) Consider the function
$f(x,y) = \ln(2+x+y)e^{x-y} - \ln(2)$.

a) Compute the partial derivatives f_x and f_y.

$$\begin{aligned} f_x(x,y) &= \frac{e^{x-y}}{2+x+y} + \ln(2+x+y)e^{x-y} \\ f_y(x,y) &= \frac{e^{x-y}}{2+x+y} - \ln(2+x+y)e^{x-y} \end{aligned}$$

b) Evaluate these partial derivatives at $(x,y) = (0,0)$.

$f_x(0,0) = (1/2) + \ln(2) \approx 1.2$, $f_y(0,0) = (1/2) - \ln(2) \approx -0.2$.

Bonus (4 pts): Use these partial derivatives to approximate $f(-0.01, 0.02)$. You can use the approximation $\ln(2) \approx 0.7$.

$f(-0.01, 0.02) \approx f(0,0) - 0.01 f_x(0,0) + 0.02 f_y(0,0) \approx 0 - 0.012 - 0.004 = -0.016$. The actual value is around -0.01565.

6. (16 pts) Double integrals. Let R be the region in the x-y plane bounded by the curve $y = \ln(x)$, the line $x = 1$ and the line $y = 2$.

[Note: All of this problem was done either on the first midterm or in class.]

a) Compute the area of R. (There are several ways to do this.)

Slicing horizontally gives $\int_0^2 (e^y - 1)dy = e^y - y|_0^2 = e^2 - 3$. Slicing vertically gives $\int_1^{e^2} (2 - \ln(x))dx = 3x - x\ln(x)|_1^{e^2} = e^2 - 3$, where you need to integrate by parts to get $\int \ln(x)dx$.

b) Convert the double integral $\iint_R 2\pi x dA$ into an iterated integral where you integrate first over y and then over x. Be explicit with your limits of integration!

Viewing this as a Type I integral, we get

$$\int_{x=1}^{e^2} \int_{y=\ln(x)}^{2} 2\pi x dy dx$$

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c) Convert the double integral $\iint_R 2\pi x dA$ into an iterated integral where you integrate first over x and then over y. As with the previous part, be explicit with your limits of integration.

As a Type II integral, it is

$$\int_{y=0}^{2} \int_{x=1}^{e^y} 2\pi x dx dy.$$

d) Evaluate $\iint_R 2\pi x dA$ by whichever method you prefer.

The second integral is easier, giving $\int_0^2 \pi x^2|_1^{e^y} dy = \pi \int_0^2 e^{2y} - 1 dy = \pi(e^4 - 5)/2$. (The first integral requires integration by parts, and of course gives the same answer.)

There were also 6 Quest problems, each worth 5 points and available as a separate file. The answers to these questions were 6, 6, 3, 3, 2 and 4.