M408M Final Exam Solutions, December 14, 2013

1. Lines and planes. (16 pts) Let $P(1,0,1)$, $Q(0,1,2)$, $R(-1,-1,1)$, and $S(0,0,10)$ be points in $\mathbb{R}^3$.

a) Let L be the line through P and Q. Express the equation of L in either vector or parametric form (your choice), and then express the equation in symmetric form.

We first compute $\overrightarrow{PQ} = \langle -1, 1, 1\rangle$. Our line in vector form is then $\mathbf{r(t)} = P + t\overrightarrow{PQ} = \langle 1,0,1\rangle + t\langle -1,1,1\rangle$. In symmetric form this works out to $1-x = y = z-1$.

b) Find the equation of the plane through P, Q and R.

First we compute the normal vector $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1,1,1\rangle \times \langle -2,-1,0\rangle = \langle 1,-2,3\rangle$. Our plane is then $1(x-1) - 2y + 3(z-1) = 0$, or $x - 2y + 3z = 4$.

c) Find the distance from S to the plane you found in part (b).

Since $\overrightarrow{PS} = \langle -1, 0, 9\rangle$, the distance is $|\overrightarrow{PS} \cdot \mathbf{n}|/|\mathbf{n}| = 26/\sqrt{14}$.

d) Find the distance from R to the line L.

This is $|\overrightarrow{PR} \times \overrightarrow{PQ}|/|\overrightarrow{PQ}| = |\langle -1,2,-3\rangle|/|\langle -1,1,1\rangle| = \sqrt{14/3}$.

2. Parametric curves (12 pts)

a) First consider the parametric curve $x = 2t - 2\sin(t)$, $y = 3 - 2\cos(t)$ in $\mathbb{R}^2$. Find the slope of the line tangent to this curve at $t = \pi/4$ (i.e. tangent at the point $(\pi/2 - \sqrt{2}, 3 - \sqrt{2})$).

At $t = \pi/4$, $dy/dt = 2\sin(t) = \sqrt{2}$ and $dx/dt = 2 - 2\cos(t) = 2 - \sqrt{2}$, so the slope is $dy/dx = \frac{dy/dt}{dx/dt} = \frac{\sqrt{2}}{2-\sqrt{2}} = 1 + \sqrt{2}$.

b) Now consider the curve $\mathbf{r}(t) = \langle 2t - 2\sin(t), 3 - 2\cos(t), 4t\rangle$. If this is the trajectory of a particle, find the velocity and acceleration as a function of time.

$\mathbf{v} = \langle 2 - 2\cos(t), 2\sin(t), 4\rangle$ and $\mathbf{a} = \langle 2\sin(t), 2\cos(t), 0\rangle$.

c) Find the equation of the line tangent to this 3D curve at $t = \pi/4$. (You can express your answer in your choice of vector, parametric, or symmetric form.)

The tangent vector to the line is $\mathbf{v}(\pi/4) = \langle 2-\sqrt{2}, \sqrt{2}, 4\rangle$. In vector form, the line is then $\mathbf{r}(t) = \langle \frac{\pi}{2} - \sqrt{2}, 3 - \sqrt{2}, \pi\rangle + t\langle 2 - \sqrt{2}, \sqrt{2}, 4\rangle$.

3. Polar coordinates. (16 pts) Consider the polar curve $r = e^\theta$, where θ runs from 0 to 2π.

a) Find the slope of this curve at $\theta = \pi/6$.

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos(\theta) + r'\sin(\theta)}{-r\sin(\theta) + r'\cos(\theta)} = \frac{e^\theta\cos(\theta) + e^\theta\sin(\theta)}{e^\theta\cos(\theta) - e^\theta\sin(\theta)}$. Plugging in $\cos(\pi/6) = \sqrt{3}/2$ and $\sin(\pi/6) = 1/2$ gives $\frac{dy}{dx} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}$.

b) Find the arc-length of this curve.

$\int_0^{2\pi} \sqrt{r^2 + (r')^2}\, d\theta = \int_0^{2\pi} \sqrt{2} e^\theta d\theta = \sqrt{2}(e^{2\pi} - 1)$.

c) Let R be the region bounded by the positive x axis, the positive y axis, and this curve. Find the area of R.

The area is $\int_0^{\pi/2} \frac{r^2}{2} d\theta = \int_0^{\pi/2} \frac{e^{2\theta}}{2} d\theta = \frac{e^{2\theta}}{4}\Big|_0^{\pi/2} = (e^\pi - 1)/4$.

d) Compute $\iint_R 3re^{-3\theta}dA$.

This is $\int_0^{\pi/2}\int_0^{e^\theta} 3re^{-3\theta} r dr d\theta = \int_0^{\pi/2} e^{-3\theta} r^3|_0^{e^\theta} d\theta = \int_0^{\pi/2} d\theta = \pi/2$.

4. Partial derivatives and gradients. (16 pts) Let $f(x, y, z) = y^2 + xy + (y + 1)\ln(z) + 3y$.

a) Compute the gradient of $f(x, y, z)$ as a function of (x, y, z).

$\vec{\nabla} f = \langle y, 2y + x + \ln(z) + 3, \frac{y+1}{z}\rangle$.

b) Compute all the second order partial derivatives. (There are 9 of these, but $f_{xy} = f_{yx}$, etc, leaving six calculations.)

$f_{xx} = 0$, $f_{xy} = f_{yx} = 1$, $f_{xz} = f_{zx} = 0$, $f_{yy} = 2$, $f_{yz} = f_{zy} = \frac{1}{z}$ and $f_{zz} = \frac{-(y+1)}{z^2}$.

c) If $\mathbf{r}(t) = \langle\cos(t), \sin(t), 1 + t\rangle$ is a parametrized curve, compute $\frac{df(\mathbf{r}(t))}{dt}$ at $t = 0$.

At $t = 0$, $x = 1$, $y = 0$ and $z = 1$, so the gradient is $\langle 0, 4, 1\rangle$. Meanwhile $d\mathbf{r}/dt = \langle -\sin(t), \cos(t), 1\rangle = \langle 0, 1, 1\rangle$. $df(\mathbf{r})/dt = \vec{\nabla} f \cdot \frac{d\mathbf{r}}{dt} = 5$.

d) Let S be the surface $f(x, y, z) = 0$. Find the equation of the plane tangent to S at $(1, 0, 1)$.

The normal vector is the gradient, so our tangent plane is $4y + z = 1$.

5. Maxima and minima. (12 pts) Note that parts (a) and (b) have nothing to do with parts (c) and (d). Parts (c) and (d) ask you to solve the same problem in two different ways. To get credit for a part, you **must** solve it using the method indicated.

a) Find all local extrema of the function $h(x, y) = x^3 - 12xy + 8y^3$.

Since $f_x = 3x^2 - 12y$ and $f_y = -12x + 24y^2$, the critical points are when $y = x^2/4$ and $x = 2y^2$, which implis that $y = y^4$, so $y = 0$ or $y = 1$. Since $x = 2y^2$, our critical points are then at $(0, 0)$ and $(2, 1)$.

b) For each critical point, indicate whether it is a local maximum, a local minimum, or a saddle point.

$f_{xx} = 6x$, $f_{xy} = 1$ and $f_{yy} = 48y$. At $(0, 0)$ we have $f_{xx} = f_{yy} = 0$, so this is a saddle point. At $(2, 1)$ we have $f_{xx} = 12$ and $f_{yy} = 48$. Since $f_{xx}f_{yy} > f_{xy}^2$, this is a local minimum.

c) **Using Lagrange multipliers**, find the maximum and minimum values of $f(x, y) = xy$ on the ellipse $\frac{x^2}{4} + y^2 = 1$.

$\vec{\nabla} f = \langle y, x\rangle$ and $\vec{\nabla} g = \langle x/2, 2y\rangle$, so our equations are $y = \lambda x/2$ and $x = 2\lambda y$. Combining these gives $x = \lambda^2 x$, so either $x = 0$ or $\lambda = \pm 1$. $x = 0$ doesn't work, since then $y = 0\lambda/2 = 0$, and we're not on the ellipse. So we must have $\lambda = \pm 1$. If $\lambda = 1$, then $x = 2y$, and (using the fact that $x^2/4 + y^2 = 1$) we're at either $(\sqrt{2}, \sqrt{2}/2)$ or $(-\sqrt{2}, -\sqrt{2}/2)$, both of which have $xy = 1$. If $\lambda = -1$, the we're at either $(\sqrt{2}, -\sqrt{2}/2)$ or $(-\sqrt{2}, \sqrt{2}/2)$, both of which have $xy = -1$. So the maximum value is 1 and the minimum value is -1.

d) Now write the ellipse as a **parametrized curve** and express $f(x, y)$ as a function of t. Then use 1-dimensional calculus to find the maximum and minimum values of f.

The simplest parametrization of the ellipse is $x = 2\cos(t)$ and $y = \sin(t)$. Then $f(x, y) = 2\sin(t)\cos(t) = \sin(2t)$. This reaches a maximum value of 1 at $2t = \pi/2$ (so $t = \pi/4$ or $5\pi/4$) and a minimum value of -1 at $2t = -\pi/2$ (so $t = 3\pi/4$ or $-\pi/4$). Those four times correspond exactly to the four points we found earlier.

6. (4 pts) Let R be the region in the plane between the parabola $y = x^2$ and the line $y = 2x$. Compute $\iint_R x^2 + 2y dA$.

Integrating first over y and then x, this is $\int_0^2 \int_{x^2}^{2x} x^2 + 2y dy dx = \int_0^2 x^2 y + y^2 |_{x^2}^{2x} dx = \int_0^2 2x^3 + 4x^2 - 2x^4 dx = 8 + \frac{32}{3} - \frac{64}{5} = \frac{88}{15}$

7. A trickier double integral. (12 pts) Consider the iterated integral

$$\int_0^1 \int_{x^2}^1 \sin(\pi y^{3/2}) dy dx.$$

a) Draw the region of integration. (In other words, rewrite the iterated integral as a double integral $\iint_R$ (some function)dA and draw a picture of R.)

This is the region in the first quadrant above the parabola $y = x^2$, and below the line $y = 1$. Note that its left boundary is the y-axis and its right boundary is $y = x^2$, or equivalently $x = \sqrt{y}$.

b) Rewrite the double integral as an iterated integral where we first integrate over x and then over y. Be careful with your limits of integration!

$$\int_{y=0}^1 \int_{x=0}^{\sqrt{y}} \sin(\pi y^{3/2}) dx dy.$$

c) Evaluate this new-and-improved iterated integral.

After doing the inner integral, we have $\int_0^2 \sqrt{y} \sin(\pi y^{3/2}) dy = -\frac{2\cos(\pi y^{3/2})}{3\pi}\Big|_0^1 = \frac{4}{3\pi}$.

8. Change of variables. (12 pts) Let R be the diamond-shaped region with corners at $P(1, 0)$, $Q(2, -1)$, $S(3, 0)$ and $T(2, 1)$. We are trying to evaluate the double integral $\iint_R e^{x+2y} dA$.

a) Write x and y as functions of new parameters u and v, so that we are at P when $u = v = 0$, Q when $u = 1$ and $v = 0$, T when $v = 1$ and $u = 0$, and S when $u = v = 1$. In other words, find a mapping that sends the unit square to R.

As we discussed in class and in the review session, we want $\langle x, y\rangle = P + u\overrightarrow{PQ} + v\overrightarrow{PT} = \langle 1, 0\rangle + u\langle 1, -1\rangle + v\langle 1, 1\rangle$. Breaking things up into components, this is $x = 1+u+v$, $y = v-u$.

b) Find the Jacobian of this mapping.

The matrix of partial derivatives is $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, whose determinant is 2.

c) Rewrite the double integral as an integral over u and v.

Since $dxdy = 2dudv$ and $e^{x+2y} = e^{1-u+3v}$, we have

$$\int_0^1 \int_0^1 2e^{1-u+3v} du dv.$$

d) Evaluate the double integral.

This works out to $\frac{2}{3}(e^3 - 1)(e - 1) = \frac{2}{3}(e^4 - e^3 - e + 1)$.