M408S First Midterm Exam Solutions, February 21, 2013

1 (12 pts) A person drives her car from noon to 4PM at a speed of $50 + 4t + 10(\sin(\pi t))$ miles per hour, where t is the number of hours since noon. (I.e., $t = 2.5$ means 2:30 PM). At 1PM, the odometer reads 29850 miles. What does the odometer read at 3PM? (Yes, you can assume that the odometer is accurate!)

The change in odometer reading is the definite integral of the speed, namely

$$\int_1^3 50{+}4t{+}10\sin(10t)dt = 50t + 2t^2 - \frac{10}{\pi}\cos(\pi t)\Big|_1^3 = (150{+}18{+}\frac{10}{\pi}){-}(50{+}2{+}\frac{10}{\pi}) = 116,$$

so the final odometer reading is 29850+116=29966.

2) (10 pts) a) Compute $\sin(\tan^{-1}(3))$

Draw a right triangle with an angle with opposite side 3, adjacent side 1, and hypotenuse $\sqrt{10}$. The sine of the angle is opposite/hypotenuse $= 3/\sqrt{10}$.

(b) Compute $\cot(\sin^{-1}(3/5))$

Now the triangle has opposite side 3, hypotenuse 5, and adjacent side $\sqrt{5^2 - 3^2} = 4$. The cotangent is then $4/3$.

3) (10 pts) Compute $\int 4x\sin(2x)dx$.

Integrate by parts with $u = x$, $du = dx$, $dv = 4\sin(2x)dx$, and so $v = -2\cos(2x)$. We then have

$$\int 4x\sin(2x)dx = -2x\cos(2x) + \int 2\cos(2x)dx = -2x\cos(2x) + \sin(2x) + C.$$

4) (10 pts) Compute $\displaystyle\int \frac{dx}{(1+x^2)^{3/2}}$

Let $x = \tan(\theta)$, so $dx = \sec^2(\theta)d\theta$ and $1 + x^2 = \sec^2(\theta)$. The integral becomes

$$\int \frac{\sec^2(\theta)d\theta}{\sec^3(\theta)} = \int \cos(\theta)d\theta = \sin(\theta) + C = \frac{x}{\sqrt{x^2+1}} + C.$$

(To get $\sin(\theta)$, draw a triangle with sides 1 and x and hypotenuse $\sqrt{1+x^2}$.)

5) (10 pts) Compute $\displaystyle\int_0^{\pi/2} \sin^3(x)\cos^3(x)dx$.

Since the power of $\cos(x)$ is odd, we can write

$$\begin{aligned}\int_0^{\pi/2} \sin^3(x)\cos^3(x)dx &= \int_0^{\pi/2} \sin^3(x)(1-\sin^2(x))\cos(x)dx \\ &= \int_{x=0}^{\pi/2} u^3 - u^5 du \\ &= \left.\frac{\sin^4(x)}{4} - \frac{\sin^6(x)}{6}\right|_0^{\pi/2} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12},\end{aligned}$$

where we used $u = \sin(x)$. This problem can also be done by converting $\sin^2(x)$ to $1-\cos^2(x)$ and using $u = \cos(x)$.

6) (10 pts) Compute $\displaystyle\int_2^3 \frac{2dx}{x^3 - x}$

This is partial fractions. Since $x^3 - x = x(x+1)(x-1)$, we have $\frac{2}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$. Clearing denominators we get $2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$. Plugging in $x = 0$, $x = 1$ and $x = 2$ gives $A = -2$, $B = 1$ and $C = 1$. So our integral is

$$\begin{aligned}\int_2^3 \frac{-2}{x} + \frac{1}{x+1} + \frac{1}{x-1}dx &= -2\ln(x) + \ln(x+1) + \ln(x-1)|_2^3 \\ &= 3\ln(2) + \ln(4) - 3\ln(3) = 5\ln(2) - 3\ln(3),\end{aligned}$$

or equivalently $\ln(32/27)$. (We didn't need absolute values since our terms are all positive.)

7) (10 pts) Compute $\displaystyle\int \frac{2e^{2x} + \sec^2(x)}{(e^{2x} + \tan(x))^2}dx$

If we set $u = e^{2x} + \tan(x)$, then we have $\int \frac{du}{u^2} = \frac{-1}{u} + C = \frac{-1}{e^{2x}+\tan(x)} + C$.

8) (10 pts) For what real values of p does the integral $\int_1^\infty \frac{dx}{x^p}$ converge? For what values does $\int_0^1 \frac{dx}{x^p}$ converge? For what values does $\int_0^\infty \frac{dx}{x^p}$ converge? Be sure to justify your answers.

Since the indefinite integral of x^{-p} is $x^{1-p}/(1-p)$ (except for $p = 1$, when we get a log), the integral from 1 to ∞ converges when $p > 1$, since $\lim_{b\to\infty} b^{1-p} = 0$ when $1-p < 0$. Similarly, the integral from 0 to 1 converges when $p < 1$, since then we have a positive power of b. The integral from 0 to ∞ never converges. (When $p = 1$ we get $\ln(x)$, which blows up at both 0 and ∞.)

9 (18 pts) Let R_1 be the region between the curve $y = \ln(x+1)$, the x axis, and the line $x = e^2 - 1$. Let R_2 be the region between the curve $y = \ln(x+1)$,

the y axis, and the line $y = 2$. In this problem you get part credit for setting up the integrals correctly and full credit for setting them up and evaluating them correctly.

a) Compute the area of R_1 by slicing vertically.

This is $\int_0^{e^2-1} \ln(x+1)dx$. This can be done either with the u-substitution $u = x + 1$ (turning it into $\int \ln(u)du = u\ln(u) - u + C$) or by integrating by parts with $u = \ln(x + 1)$, $dv = dx$, and $v = x + 1$, or (slightly more complicated) by integrating by parts with $u = \ln(x + 1)$, $v = x$. The answer is $(x + 1)\ln(x + 1) - (x + 1)|_0^{e^2-1} = e^2 + 1$.

b) Compute the area of R_2 by slicing horizontally.

If $y = \ln(x + 1)$, then $e^y = x + 1$, so $x = e^y - 1$. This gives the integral

$$\int_0^2 e^y - 1dy = e^y - y|_0^2 = e^2 - 3.$$

c) Find the volume of the region obtained by rotating R_2 around the y axis.

By disks, the volume is

$$\begin{aligned}\pi\int_0^2 (e^y - 1)^2 dy &= \pi\int_0^2 e^{2y} - 2e^y + 1dy \\ &= \pi\left(\frac{e^{2y}}{2} - 2e^y + y\right)\Bigg|_0^2 \\ &= \pi(\frac{e^4}{2} - 2e^2 + 2 - (\frac{1}{2} - 2)) = \pi\left(\frac{e^4}{2} - 2e^2 + \frac{7}{2}\right)\end{aligned}$$