M408S Second Midterm Exam, March 21, 2013

1 (30 points) This problem is about methods of integration. For each integral, you need to explain which technique you would use to do the integral, and provide a few details to back that up. You do **not** have to finish the computation. If an integral can be solved in multiple ways, then either correct answer will be accepted.

If you say "u-substitution", indicate what u and du are, and rewrite the integral in terms of u. (Then STOP.)

If you say "integrate by parts", indicate what u and v (not just u and dv) are. (Then STOP.)

If you say "trig integral", indicate what trig identities you are using, and how. E.g. you might say "use $\sec^2(x) = 1 + \tan^2(x)$ to convert all but two of the secants to cosines so we can do a u-sub with $u = \tan(x)$". (Then STOP.)

If you say "trig substitution", indicate which substitution you are using and draw the relevant triangle. (Then STOP. Do not attempt the resulting trigonometric integral.)

If you say "partial fractions", write the integrand as a sum of terms with unknown coefficients. E.g., you might say "convert the integrand to $\frac{A}{x-3} + \frac{B}{x+7}$." (Then STOP. You do NOT have to compute the coefficients A, B, etc. or do the resulting integrals.)

If an integral can't be done in any of these ways, say "none of these".

a) $\displaystyle\int \frac{2x\sec^2(\ln(x^2+5))}{x^2+5}dx$

This is a u-substitution with $u = \ln(x^2+5)$ and $du = 2x/(x^2+5)$, making the integral $\int \sec^2(u)du$.

b) $\displaystyle\int \frac{3x^3 + 173x^2 + 4x - 314159}{x^4 + 4x^2}dx$

This is partial fractions. Convert the integrand to $\frac{Ax+B}{x^2+4} + \frac{C}{x} + \frac{D}{x^2}$.

c) $\displaystyle\int (x^2-9)^{5/2}dx$

This is a trig substitution with $x = 3\sec(\theta)$. The triangle should have a hypotenuse of x, and adjacent side of 3, and an opposite side of $\sqrt{x^2-9}$.

d) $\displaystyle\int x^2\ln(x)dx$

Integrate by parts with $u = \ln(x)$, $dv = x^2dx$ and $v = x^3/3$.

e) $\int \sin^4(x)\cos^7(x)dx$

Trig integral. Use $\cos^2(x) = 1 - \sin^2(x)$ to convert all but one of the cosines into sines and then do a u-sub with $u = \sin(x)$.

2) (20 points) Consider the differential equation $\frac{dy}{dx} = y\cos(x)$.

(a) Find the general solution to this differential equation.

Since $dy/y = \cos(x)dx$, we have $\ln(|y|) = \sin(x) + C$, so $y = Ae^{\sin(x)}$ for some unknown constant A.

b) If $y(0) = 5$, what is $y(\pi/2)$?

Since $\sin(0) = 0$ and $e^0 = 1$, we get $A = 5$, so $y(\pi/2) = 5e^{\sin(\pi/2)} = 5e$.

3) (20 points) [Please give exact answers. You can leave them in terms of exponentials and logs. E.g., an answer might be $5\ln(1/3) - e^{0.1}$. Except that isn't actually a correct answer.] A colony of bacteria is growing exponentially. At noon there are 1,000 bacteria. An hour later, there are 3,000 bacteria.

a) How long does it take for the population to double?

For exponential growth we have $y(t) = y(0)e^{kt}$. Since $y(1) = 3y(0)$, we have $e^k = 3$, so $k = \ln(3)$. Now, for doubling we want $y(t) = 2y(0)$, or $e^{kt} = 2$, so $kt = \ln(2)$, so our doubling time is $t = \ln(2)/k = \ln(2)/\ln(3)$.

b) Find the population t hours after noon, as a function of t.

We essentially did that in the solution to (a). The answer is $y(t) = 1,000e^{\ln(3)t}$.

c) At what time will the population hit 5,000?

Solving $5000 = 1000e^{kt}$ we get $kt = \ln(5)$, so $t = \ln(5)/\ln(3)$.

4)(20 points) a) Does $\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n}$ converge? If so, to what? If not, why not?

This is a difference of two convergent geometric series, so it converges. $\sum_{n=0}^{\infty} 3^n/4^n = 1/(1-(3/4)) = 4$ and $\sum_{n=0}^{\infty} 2^n/4^n = 1/(1-(1/2)) = 2$, so the total is 2.

b) Does the sequence $\{n\sin(2\pi/n)\}$ converge? If so, to what? If not, why not?

It converges to 2π. There are two ways to see this. One is to use L'Hospital's rule on $n\sin(2\pi/n) = \frac{\sin(2\pi/n)}{1/n}$. The other is to remember that $\sin(x) \approx x$ for x small. Here $x = 2\pi/n$, so our expression is roughly $n(2\pi/n) = 2\pi$.

c) Does the series $\sum_{n=1}^{\infty} n\sin(2\pi/n)$ converge? If so, why? If not, why not? (You are **not** expected to evaluate the sum.)

Since the terms are not approaching 0 (by part b they are approaching 2π), the sum diverges.

5) (10 points) Indicate whether each of these integrals converges, and why it does or doesn't. You don't need to say what it converges to.

a) $\displaystyle\int_0^\infty 5xe^{-x^2}dx$

After the u-substitution $u = x^2$ this becomes $\int_0^\infty (5/2)e^{-u}du$, which converges. In fact it converges to $5/2$.

b) $\displaystyle\int_0^\pi \sec^2(x)dx$

This diverges. The integrand blows up at $x = \pi/2$, so we have to break this into two pieces, $\int_0^{\pi/2} \sec^2(x)dx$ and $\int_{\pi/2}^{\pi} \sec^2(x)dx$. Each of these pieces is divergent, since $\tan(x)$ blows up as $x \to \pi/2$ from either direction.