M408M Final Exam Solutions, December 9, 2015

1) A polar curve. Let C be the portion of the "cloverleaf" curve $r = \sin(2\theta)$ that lies in the first quadrant.

a) Draw a rough sketch of C.

This looks like one quarter of a cloverleaf. The curve is tangent to both the x and y-axes, since $r = 0$ at $\theta = 0$ and $\theta = \pi/2$. In between, in bulges out, reaching a maximum radius of 1 when $\theta = \pi/4$.

b) Write down an integral that gives the arc-length of C. Simplify the integrand as much as possible, but **do not attempt to compute the integral.** (It can't be done in closed form).

The arc-length is given by $s = \int_0^{\pi/2} \sqrt{r^2 + (dr/d\theta)^2} d\theta$, which equals $\int_0^{\pi/2} \sqrt{\sin^2(2\theta) + 4\cos^2(2\theta)} d\theta = \int_0^{\pi/2} \sqrt{1 + 3\cos^2(\theta)} d\theta$. This is an elliptic integral and cannot be done in closed form.

c) Compute the area enclosed by C. (This integral **can** be done in closed form, and I expect you to do it.)

$$\int_0^{\pi/2} \frac{r^2}{2} d\theta = \int_0^{\pi/2} \frac{1}{2}\sin^2(2\theta) d\theta = \int_0^{\pi/2} \frac{1}{4}(1 - \cos(4\theta)) d\theta = \frac{\theta}{4} - \frac{\sin(4\theta)}{16}\Big|_0^{\pi/2} = \frac{\pi}{8}.$$

2. Lines and planes. (2 pages!) Let $P(3, -1, 4)$, $Q(2, 1, 7)$, and $R(1, 5, 8)$ be points in $\mathbb{R}^3$. Let $\mathcal{L}$ be the line through P and Q, and let $\mathcal{T}$ be the plane containing all three points.

a) Give a parametrization for $\mathcal{L}$.

Since $\vec{PQ} = \langle -1, -2, 3 \rangle$, we have $\mathbf{r}(t) = \langle 3, -1, 4 \rangle + \langle -1, 2, 3 \rangle t$. (That's in vector form. In coordinates, that would be $x(t) = 3 - t$, $y(t) = -1 + 2t$, $z(t) = 4 + 3t$.)

b) Write down the symmetric equations for $\mathcal{L}$. (That is, the equations relating x, y and z that don't involve the parameter t.)

$$\frac{x-3}{-1} = \frac{y+1}{2} = \frac{z-4}{3}.$$

c) Find a vector normal to $\mathcal{T}$.

This is

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 3 \\ -2 & 6 & 4 \end{vmatrix} = \langle -10, -2, -2 \rangle.$$

You could also rescale this to $\langle 5, 1, 1 \rangle$.

d) Find the equation of $\mathcal{T}$. Simplify as much as possible.

$5(x-3)+(y+1)+(x-3)=0$, or $5x+y+z=18$.

3. Curves. Consider the curve $\mathbf{r}(t) = (1+t^2, 3-2\ln(t), 5+2\sqrt{2}(t-1))$.

a) Find the arc-length of the curve traced out as t goes from 1 to 3.

Since the velocity is $\langle 2t, -2/t, 2\sqrt{2} \rangle$, the speed is $\sqrt{4t^2+4/t^2+8} = 2t + 2/t$, and the arc-length is $\int_1^3 (2t+2/t)dt = t^2 + 2\ln(t)\Big|_1^3 = 8 + 2\ln(3)$.

b)When $t=1$, this curve goes through the point $P(2,3,5)$. Find the tangent, principal normal, and binormal vectors at this point.

The velocity is $\langle 2t, -2/t, 2\sqrt{2} \rangle = \langle 2, -2, 2\sqrt{2} \rangle$, so the tangent is $\mathbf{T} = \frac{\langle 1, -1, \sqrt{2} \rangle}{2}$.

The acceleration is $\langle 2, 2/t^2, 0 \rangle = \langle 2, 2, 0 \rangle$. Note that this is orthogonal to the velocity, so it points in the direction of the principal normal, and $\mathbf{N} = \frac{\langle 1,1,0 \rangle}{\sqrt{2}}$. Then $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{\langle -1, 1, \sqrt{2} \rangle}{2}$.

If you didn't notice that the acceleration was orthogonal to the velocity, you could compute $\mathbf{B}$ from $\mathbf{v} \times \mathbf{a}$ and then compute $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

4. Consider the function $f(x,y) = xy^3 - x^2y$.

a) Compute the partial derivatives f_x and f_y (as functions of x and y).

$f_x = y^3 - 2xy$ and $f_y = 3xy^2 - x^2$.

b) The surface $z = f(x,y)$ contains the point $(-3, 2, -42)$. Find the equation of the plane tangent to the surface at this point.

Evaluating at $(-3,2)$ gives $f_x = 20$ and $f_y = -45$, so $z + 42 = 20(x+3) - 45(y-2)$, or $z = 20x - 45y + 108$, or $-20x + 45 + z = 108$. You could also get this from the normal vector being $\langle -f_x, -f_y, 1 \rangle$

c) Using linearization, differentials, or the answer to (b) (all of which amount to essentially the same thing), approximate $f(-2.999, 2.002)$.

$z + 42 \approx 20(0.001) - 45(0.002) = -0.07$, so $f(-2.999, 2.002) = z \approx -42.07$.

5. Level surfaces. Consider the surface $e^x + 2y + y\ln(z) = 7$. This goes through the point (0,3,1).

a) Find a vector normal to the surface at the point (0,3,1).

$$\nabla g = \langle e^x, 2 + \ln(z), y/z\rangle = \langle 1, 2, 3\rangle.$$

b) Find the equation of the plane tangent to the surface at that point.

This comes immediately from the normal vector: $x+2(y-3)+3(z-1) = 0$, or equivalently $x + 2y + 3z = 9$.

6. Max/min. Consider the function $f(x, y) = e^y(y^2 - x^2)$.

a) Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy}.

$f_x = -2xe^y$, $f_y = (2y + y^2 - x^2)e^y$, $f_{xx} = -2e^y$, $f_{xy} = -2xe^y$, $f_{yy} = (2 + 4y + y^2)e^y$.

b) Find all the critical points of this function.

Setting $f_x = 0$ gives $x = 0$ (since e^y is never zero). Then setting $f_y = 0$ gives $2y + y^2 = 0$, hence $y = 0$ or $y = -2$. So our two critical points are $(0, 0)$ and $(0, -2)$.

c) For each critical point, use the second derivative test to determine whether the critical point is a local maximum, a local minimum, or a saddle point.

At $(0, 0)$, we have $f_{xx} = -2$, $f_{xy} = 0$ and $f_{yy} = 2$, so this is a saddle point.

At $(0, -2)$ we have $f_{xx} = -2e^{-2}$, $f_{xy} = 0$ and $f_{yy} = -2e^{-2}$, so this is a local maximum.

7. Double integrals in Cartesian coordinates. (2 pages!)

a) Compute $\iint_{D_a} \frac{3x}{y} dA$ where D_a is the region bounded by the lines $x = 0$, $x = 1$, and $y = 1$ and the curve $y = 2e^x$.

This is a type I region. Our integral is

$$\begin{aligned}
\int_0^1 \int_1^{2e^x} \frac{3x}{y} dydx &= \int_0^1 3x\ln(y)\Big|_{y=1}^{2e^x} dx \\
&= \int_0^1 3x^2 + 3x\ln(2)dx \qquad \text{since } \ln(2e^x) = x + \ln(2) \\
&= 1 + \frac{3\ln(2)}{2}.
\end{aligned}$$

b) Compute $\iint_{D_b} x^2 y dA$ where D_b is the region bounded by the lines $y = 0$, $y = 1$, and $x = 0$ and the curve $y = \ln(x)$.

This is a Type II region, and we should rewrite $y = \ln(x)$ as $x = e^y$.

$$\begin{aligned}
\int_0^1 \int_0^{e^y} x^2 y dx dy &= \int_0^1 \frac{y e^{3y} dy}{3} \\
&= \frac{y e^{3y}}{9} - \frac{e^{3y}}{27}\Big|_0^1 \qquad \text{Integrating by parts} \\
&= \frac{e^3}{9} - \frac{e^3}{27} + \frac{1}{27} = \frac{2e^3 + 1}{27}.
\end{aligned}$$

c) Rewrite the iterated integral $\int_1^3 \int_{x^2}^{4x-3} \cos(x^2 y^3) dy\, dx$ as an iterated integral $dxdy$. (That is, swap the order of integration.) You do **NOT** need to evaluate the resulting iterated integral!

The region of integration is bounded by the curves $y = x^2$ (aka $x = \sqrt{y}$) and $y = 4x - 3$ (aka $x = (y+3)/4$), which intersect at the points $(1, 1)$ and $(3, 9)$. If we view this as a type II region, we integrate dx from $x = (y+3)/4$ to $\sqrt{y}$, and then integrate dy from $y = 1$ to $y = 9$. That is, our integral is

$$\int_1^9 \int_{\frac{y+3}{4}}^{\sqrt{y}} \cos(x^2 y^3) dx\, dy.$$

Note that the integrand has NOTHING to do with the process of switching the order of integration. It just comes along for the ride.

8. Laminae. Suppose we have a fan blade in the shape of the region you considered in problem 1. That is, it is bounded by the polar curve $r = \sin(2\theta)$ in the first quadrant. [Note that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.] The density of this blade is given by $\rho(x, y) = \frac{x}{x^2+y^2}$. [Yes, the density blows up as we approach the origin, but all of the integrals in this problem still converge. Also, this problem is best done in polar coordinates.]

a) Compute the mass of the blade.

We want to compute $\iint_D \rho dA$. In polar coordinates, $\rho = x/r^2 = \cos(\theta)/r$ and $dA = r dr\, d\theta$, so our integral is

$$\int_0^{\pi/2} \int_0^{2\sin(\theta)\cos(\theta)} \cos(\theta) dr\, d\theta = \int_0^{\pi/2} 2\sin(\theta)\cos^2(\theta) d\theta = 2/3,$$

where we did a u-substitution with $u = \cos(\theta)$, $du = -\sin(\theta) d\theta$.

b) Compute the moment of inertia I_0. [The last step in the integral is a little tricky. Remember that $\sin^2(\theta)+\cos^2(\theta)=1$.]

This is the same integral, only with an extra factor of r^2, since $I_0 = \iint_D r^2 \rho dA$. This gives

$$\begin{aligned}\int_0^{\pi/2}\int_0^{2\sin(\theta)\cos(\theta)} r^2\cos(\theta)dr\,d\theta &= \int_0^{\pi/2}\frac{8}{3}\sin^3(\theta)\cos^4(\theta)d\theta \\ &= \int_0^{\pi/2}\frac{8}{3}\sin(\theta)[\cos^4(\theta)-\cos^6(\theta)]d\theta \\ &= \frac{8}{3}\left(\frac{1}{5}-\frac{1}{7}\right)=\frac{16}{105},\end{aligned}$$

where we used $\sin^2(\theta)=1-\cos^2(\theta)$ to get to the second line and $u=\cos(\theta)$ to get to the third.

9. Mappings. (2 pages!) Let D be the parallelogram (actually a square) whose corners are $(0,0)$, $(3,-1)$, $(1,3)$ and $(4,2)$. Our goal is to compute $\iint_D e^{(x+y)/2}dA$.

a) Find a mapping that sends the unit square to D.

$\langle x,y\rangle = u\langle 3,-1\rangle + v\langle 1,3\rangle$, or $x=3u+v$ and $y=-u+3v$.

b) Rewrite our integral as an integral over the unit square. Don't forget the Jacobian!

Our integrand is $\exp(\frac{x+y}{2}) = \exp(u+2v)$ and our Jacobian is $\begin{vmatrix} 3 & 1 \\ -1 & 3\end{vmatrix} = 10$, so the integral is

$$\int_0^1\int_0^1 10e^{u+2v}dudv$$

c) Evaluate that new-and-improved integral.

Integrating over u gives $\int_0^1 10(e-1)e^{2v}dv$, and then integrating over v gives $5(e-1)(e^2-1)$, which you can also expand as $5e^3-5e^2-5e+5$.