M408M First Midterm Exam Solutions, October 8, 2015

1) (30 points, 2 pages) A case study in ellipses. Let $C$ be the ellipse whose equation in Cartesian coordinates is

$$\frac{x^2}{4} + y^2 = 1.$$

a) Sketch the ellipse.

*This is a "short and fat" ellipse, with center at the origin, vertices at* $(\pm 2, 0)$ *and* $(0, \pm 1)$. *We'll later see that the foci are at* $(\pm\sqrt{3}, 0)$ *and the directrices are at* $x = \pm 4/\sqrt{3}$.

b) Find a parametrization of this ellipse. (There is more than one right answer, by the way.)

*The simplest one is* $x = 2\cos(t)$, $y = \sin(t)$, *with* $t$ *ranging from 0 to* $2\pi$.

c) Using your result from (b), write down an integral that equals the perimeter of the ellipse. Simplify your integrand as much as possible, but **do not attempt to compute the integral!** (This is called an "elliptic integral" and cannot be computed in closed form.)

*The speed is* $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{4\sin^2(t) + \cos^2(t)} = \sqrt{1 + 3\sin^2(t)}$, *so the perimeter is* $\int_0^{2\pi} \sqrt{1 + 3\sin^2(t)}\, dt$.

d) Find the locations of the foci of the ellipse, and mark the foci on your sketch.

*Since* $c^2 = a^2 - b^2 = 4 - 1 = 3$, *the foci are on the major axis, a distance* $\sqrt{3}$ *from the center, i.e. at* $(\pm\sqrt{3}, 0)$.

e) Compute the eccentricity of the ellipse.

$e = c/a = \sqrt{3}/2$

f) Find the equation of a directrix of the ellipse, and draw the directrix in your sketch. (There are two directrices, one for each focus. You just need to find one of them.)

*The directrix is a vertical line* $x = x_0$. *Since the distance from the point* $(2, 0)$ *to the focus* $(\sqrt{3}, 0)$ *is* $e$ *times the distance to the directrix, we must have* $2 - \sqrt{3} = e(x_0 - 2) = \sqrt{3}(x_0 - 2)/2$. *A little algebra gives* $x_0 = 4/\sqrt{3}$. *By symmetry, the other directrix is at* $x = -4/\sqrt{3}$.

2. (25 points, 2 pages) Polar coordinates. Let $S$ be the polar curve $r = 2\cos(\theta)$, and let $C$ be the circle $r = 1$. (Note: if you get stuck on (a) or (b), don't give up. You may still be able to work (c), (d), and (e).)

a) Sketch the curve $S$.

*This is an example we already did in class. It is a circle of radius 1 centered at* $(1, 0)$.

b) Find the equations of $S$ in Cartesian coordinates.

*Since* $r = 2\cos(\theta)$, $r^2 = 2r\cos(\theta) = 2x$, *so* $x^2 + y^2 = 2x$ *(a perfectly valid answer), or equivalently* $(x-1)^2 + y^2 = 1$.

c) Find the slope of the line tangent to $S$ at $\theta = \pi/4$.

*Since* $x = r\cos(\theta) = 2\cos^2(\theta)$ *and* $y = r\sin(\theta) = 2\sin(\theta)\cos(\theta)$, *we compute* $dy/dx = \dot{y}/\dot{x} = 2(\cos^2(\theta) - \sin^2(\theta))/(-4\cos(\theta)\sin(\theta)$. *When* $\theta = \pi/4$, $\sin(\theta) = \cos(\theta) = \sqrt{2}/2$, *and the numerator is zero, while the denominator is -2, so* $dy/dx = 0$.

d) Find the points where $C$ and $S$ intersect. You can express your answer either in polar or cartesian coordinates

*Since* $1 = 2\cos(\theta)$, *we must have* $\cos(\theta) = 1/2$, *so* $\theta = \pm\pi/3$. *In polar coordinates, our intersection points are* $(r, \theta) = (1, \pm\pi/3)$. *In Cartesian coordinates they are* $(x, y) = (\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$.

e) Write down an explicit integral that gives the area of the region that is inside $S$ but outside of $C$. You should simplify the integrand and specify the limits of integration, but you **do not** need to evaluate the integral.

*The area is* $\int_{-\pi/3}^{\pi/3} \frac{1}{2}(r_1^2 - r_2^2)d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4\cos^2(\theta) - 1)d\theta$. *Unlike problem 1c, this CAN be done in closed form, but is still too grungy to put on a midterm.*

3. (25 pts, 2 pages) Lines and planes.

Let $P(1, -1, 1)$, $Q(3, 1, 4)$, $R(1, 0, 3)$ and $S(5, 1, 2)$ be points in $\mathbb{R}^3$. Let $L$ be the line through $P$ and $Q$, and let $T$ be the plane through $P$, $Q$ and $R$.

a) Find the equation of $L$. Express your answer both as a parametrization, and separately as a set of equations that $x, y, z$ satisfy.

*The vector from* $P$ *to* $Q$ *is* $< 2, 2, 3 >$, *so we can either write* $\vec{\mathbf{r}}(t) =< 1, -1, 1 > + < 2, 2, 3 > t$ *or* $x = 1 + 2t$, $y = -1 + 2t$, $z = 1 + 3t$ *(parametrization) and also write* $\frac{x-1}{2} = \frac{y+1}{2} = \frac{z-1}{3}$. *(equations)*

b) Find a vector normal to the plane $T$.

*This is the cross product of $\vec{P}Q$ and $\vec{P}R$, namely $\vec{N} =< -1, 4, -2 >$. An equally good answer is $< 1, -4, 2 >$, or any other nonzero multiple of $\vec{N}$.*

c) Find the equation of the plane $T$.

*We can read that off from the form of $\vec{N}$, namely $-x + 4y - 2z = -7$, or equivalently $x - 4y + 2z = 7$. Where did the 7 come from? From evaluating $x - 4y + 2z$ at $P$, $Q$, or $R$.*

d) Find the distance from $S$ to $T$.

*Let $\vec{w} =< 4, 2, 1 >$ be the vector from $P$ to $S$. Our answer is $|\vec{N} \cdot \vec{w}|/\|\vec{N}\| = 2/\sqrt{21}$.*

4. (20 pts) Surfaces. Identify whether each of these surfaces is a hyperboloid of one sheet, a hyperboloid of two sheets, an elliptic paraboloid, a hyperbolic paraboloid, or an ellipsoid. [Hint: there is at most one of each. Also, you may want to complete some squares.] No justification needed. No penalty for guessing. 5 points for each correct answer.

*A good preliminary is completing a few squares: $x^2 - 2x = (x-1)^2 - 1$, $z^2 - 4z = (z-2)^2 - 4$, and $x^2 - 6x = (x-3)^2 - 9$.*

a) $-x^2 + 2x + 2y^2 - 4z = 0$

*Since there is no $z^2$ term, and since the $x^2$ and $y^2$ terms come with opposite signs, this is a hyperbolic paraboloid.*

b) $-x^2 + 2x + 2y^2 + z^2 - 4z = 0$

*This is $-(x-1)^2 + 2y^2 + (z-2)^2 = 3$, a hyperboloid of one sheet.*

c) $x^2 - 2x + 2y^2 + z^2 - 4z = 0$

*All squared terms have the same sign, so this is an ellipsoid. More precisely, we have $(x-1)^2 + 2y^2 + (z-2)^2 = 5$, an ellipsoid centered at (1,0,2).*

d) $-x^2 + 6x + 2y^2 + z^2 - 4z = 0$

*This is $-(x-3)^2 + 2y^2 + (z-2)^2 = -5$, a hyperboloid of two sheets.*