M408D First Midterm Exam Solutions, February 19, 2016

1) (30 pts) Strategy for integration. Indicate the key technique for evaluating the following integrals. **You do not need to actually finish the integrals**, but you have to provide a clear road map for **how** to compute them. For instance, you answer might be "u-substitution with $u = e^x$" or "integration by parts with $u = x^2$, $dv = e^x dx$", or "integrate by parts twice to go in circles" or "trig substitution with $x = 2\sin(\theta)$, or "partial fractions with $\frac{A}{x} + \frac{Bx+C}{x^2+7}$" or "reduce degree with double-angle formula", etc. Just naming a technique like "u-sub" is only worth part credit. Describing how it applies is worth full credit.

a) $\displaystyle\int \frac{4dx}{9+(x-2)^2}$

Use the trig substitution $x-2 = 3\tan(\theta)$, or equivalently $x = 2+3\tan(\theta)$. Then $9+(x-2)^2 = 9\sec^2(\theta)$, and $dx = 3\sec^2(\theta)$, and the integral reduces to $\int \frac{4}{3} d\theta$.

b) $\displaystyle\int \frac{4dx}{x^3-x^2}$

The denominator factors as $x^2(x-1)$, so this can be done with partial fractions, expressing the integrand as $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$ (times dx, of course).

c) $\displaystyle\int (x^2+5)\ln(x)dx.$

Integrate by parts with $u = \ln(x)$ and $dv = (x^2+5)dx$.

2. (30 points) Compute the following quantities:

a) $\displaystyle\int_0^1 (x^2+3)^2 dx.$

Just multiply it out. The integral is

$\int_0^1 (x^4+6x^2+9)\,dx = (x^5/5) + 2x^3 + 9x\Big|_0^1 = 11\frac{1}{5}$ or $56/5$.

b) $\displaystyle\int_0^{\pi/2} \sin^3(x)dx$

Use $\sin^2(x) = 1-\cos^2(x)$ to convert this to $\int_0^{\pi/2}(1-\cos^2(x))\sin(x)dx$, then do a u-sub with $u = \cos(x)$, $du = -\sin(x)dx$ to convert this to

$$\int_1^0 -(1-u^2)du = \int_0^1 (1-u^2)du = u - \frac{u^3}{3}\Big|_0^1 = 2/3.$$

c) $\sin(\tan^{-1}(5/3))$.

Draw a right triangle with opposite side 5, adjacent side 3, and hypotenuse $\sqrt{3^2+5^2} = \sqrt{34}$. Then the sine of the angle is opposite/hypotenuse, or $5/\sqrt{34}$.

3. (20 pts) Differential equations

a) Find the general solution to the differential equation $\frac{dy}{dx} = e^x y^2$.

This is separable.

$$\begin{aligned} \frac{dy}{y^2} &= e^x dx \\ \int \frac{dy}{y^2} &= \int e^x dx \\ \frac{-1}{y} &= e^x + C \\ y &= \frac{-1}{e^x + C} \end{aligned}$$

b) Find the solution to $\dfrac{dy}{dx} = \dfrac{5y}{x} + 8$ with initial condition $y(1) = 7$.

This is linear with $P(x) = -5/x$ and $Q(x) = 8$. The integrating factor is $I(x) = \exp(\int -5dx/x) = \exp(-5\ln(x)) = x^{-5}$. We then have

$$\begin{aligned} x^{-5}y' - 5yx^{-6} &= 8x^{-5} \\ (x^{-5}y)' &= 8x^{-5} \\ x^{-5}y &= -2x^{-4} + C \\ y &= -2x + Cx^5. \end{aligned}$$

Since $y(1) = 7$, we must have $C = 9$, so $y = -2x + 9x^5$.

4. (2 pages, 20 points) A cold beer is heating up at a rate proportional to the difference between room temperature (75 degrees Farenheit) and the temperature of the beer.

a) Write down a differential equation for the function $T(t)$ that describes the data I just gave you. Here $T(t)$ is the temperature of the beer, in degrees Farenheit, and t is the number of minutes since pouring. (Your answer may involve some unknown proportionality constant.)

Since the rate of change (i.e. dT/dt) is proportional to (i.e. some unknown constant k times) the difference $75 - T$, we have

$$\frac{dT}{dt} = k(75 - T).$$

b) Find the general solution to this equation. This may involve an unknown proportionality constant and/or an arbitrary constant of integration.

Either by u-sub with $u = T - 75$, or by treating this as a separable equation, we get

$$T(t) = 75 + Ce^{-kt}.$$

c) Now suppose that when the beer is poured, it is at 35 degrees. 10 minutes later, it is at 55 degrees. Use this information to figure out the unknown constant(s) and to determine the temperature of the beer as a function of time.

Since $T(0) = 35$, we must have $C = -40$, so $T(t) = 75 - 40e^{-kt}$. Plugging in $T(10) = 55$ gives

$$\begin{aligned} 55 &= 75 - 40e^{-10k} \\ 40e^{-10k} &= 20 \\ e^{-10k} &= 1/2 \\ -10k &= \ln(1/2) = -\ln(2) \\ k &= \ln(2)/10 \\ T(t) &= 75 - 40e^{-t\ln(2)/10} \end{aligned}$$

d) At what time will the beer be at 70 degrees?

Here's the slow-and-steady solution:

$$\begin{aligned} 70 &= 75 - 40e^{-t\ln(2)/10} \\ 40e^{-t\ln(2)/10} &= 5 \\ e^{-t\ln(2)/10} &= 1/8 \\ -t\ln(2)/10 &= \ln(1/8) = -3\ln(2) \\ t &= 30. \end{aligned}$$

The quick and less obvious method is to notice that the difference between the temperature and 75 degrees was cut in half in 10 minutes. It needs to be cut in half three times (from 35 to 55 to 65 to 70), which takes 3 times 10 minutes, or 30 minutes.