M346 Second Midterm Exam Solutions, November 9, 2004

1. Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

Since the sum of each row is 4, $\lambda_1 = 4$. Since the trace is 5, the other eigenvalue is $\lambda_2 = 1$. The eigenvectors are $\mathbf{b}_1 = (1, 1)^T$ and $\mathbf{b}_2 = (2, -1)^T$.

b) Write down the most general solution to the difference equation $\mathbf{x}(n) = A\mathbf{x}(n-1)$.

$\mathbf{x}(n) = c_1 4^n \mathbf{b}_1 + c_2 \mathbf{b}_2$, where c_1 and c_2 are arbitrary constants.

c) Find $\mathbf{x}(n)$ when $\mathbf{x}(0) = (7, 1)^T$.

Since (by row reduction, say) we have $\mathbf{x}(0) = 3\mathbf{b}_1 + 2\mathbf{b}_2$, $\mathbf{x}(n) = 3(4^n)\mathbf{b}_1 + 2\mathbf{b}_2 = \begin{pmatrix} 3 \cdot 4^n + 4 \\ 3 \cdot 4^n - 2 \end{pmatrix}$

2. Let $A = \begin{pmatrix} 3 & 2 \\ -5 & -4 \end{pmatrix}$ and consider the differential equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

a) Find the eigenvalues and eigenvectors of A.

Since the trace is -1 and the determinant is -2, the eigenvalues are $\lambda_1 = 1$ (with $\mathbf{b}_1 = (1, -1)^T$) and $\lambda_2 = -2$ (with $\mathbf{b}_2 = (2, -5)^T$).

b) How many stable modes are there, and what are they? How many unstable modes are there and what are they? How many neutrally stable modes are there and what are they? What is the dominant mode?

There is one negative eigenvalue (λ_2) hence one stable mode ($\mathbf{b}_2$). There is one positive eigenvalue (λ_1) hence one unstable mode ($\mathbf{b}_1$). There are no neutral modes. The $\mathbf{b}_1$ mode is dominant, since $\lambda_1 > \lambda_2$.

c) If $\mathbf{x}(0) = (\pi, \sqrt{17})^T$, find the limit, as $t \to \infty$, of $x_1(t)/x_2(t)$.

Asymptotically, $\mathbf{x}(t)$ will point in the dominant $\mathbf{b}_1$ direction, so the limiting ratio is $1/-1 = -1$.

3. Consider the NONLINEAR system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= 2(x_2^2 - 1) \\ \frac{dx_2}{dt} &= (x_1^2 - 1)/2 \end{aligned}$$

a) Find the fixed points (there are four of them).

Setting $x_2^1 - 1 = 0$ gives $x_2 = \pm 1$. Likewise $x_1 = \pm 1$. So the four fixed points are (1,1), (1,-1), (-1,1) and (-1,-1).

b) For each fixed point, approximate the deviations from that fixed point by a linear system of equations.

Letting $\mathbf{y} = \mathbf{x} = \mathbf{a}$, where $\mathbf{a}$ is the fixed point, we have $d\mathbf{y}/dt \approx A\mathbf{y}$, where $A = \begin{pmatrix} 0 & 4x_2 \\ x_1 & 0 \end{pmatrix}$, evaluated at the fixed point. That is,

Near $(1,1)$ we use $A = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$, whose eigenvalues are ± 2 (this is unstable).

Near $(1,-1)$ we use $A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$, whose eigenvalues are $\pm 2i$ (this is neutral, as the real parts are both zero).

Near $(-1,1)$ we use $A = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}$, whose eigenvalues are $\pm 2i$ (this is neutral).

Near $(-1,-1)$ we use $A = \begin{pmatrix} 0 & -4 \\ -1 & 0 \end{pmatrix}$, whose eigenvalues are ± 2 (this is unstable).

c) Which fixed points are stable? Unstable? Borderline?

(See answer to b)

4. In $\mathbb{R}^3$, with the usual inner product, consider the vectors $\mathbf{b}_1 = (1,2,2)^T$, $\mathbf{b}_2 = (4,4,3)^T$, $\mathbf{b}_3 = (2,9,-1)^T$.

a) Use the Gram-Schmidt process to convert this basis for $\mathbb{R}^3$ into an orthogonal basis for $\mathbb{R}^3$.

$\mathbf{y}_1 = \mathbf{b}_1 = (1,2,2)^T$.

$\mathbf{y}_2 = \mathbf{b}_2 - \frac{\langle \mathbf{y}_1 | \mathbf{b}_2 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle}\mathbf{y}_1 = \mathbf{b}_2 - 2\mathbf{y}_1 = (2,0,-1)^T$.

$\mathbf{y}_3 = \mathbf{b}_3 - \frac{\langle \mathbf{y}_1 | \mathbf{b}_3 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle}\mathbf{y}_1 - \frac{\langle \mathbf{y}_2 | \mathbf{b}_3 \rangle}{\langle \mathbf{y}_2 | \mathbf{y}_2 \rangle}\mathbf{y}_2 = \mathbf{b}_3 - 2\mathbf{y}_1 - \mathbf{y}_2 = (-2,5,-4)^T$.

b) If $\mathbf{v} = (5,2,9)^T$, compute $P_{\mathbf{b}_1}\mathbf{v}$.

$P_{\mathbf{b}_1}\mathbf{v} = \frac{\langle \mathbf{b}_1 | \mathbf{v} \rangle}{\langle \mathbf{b}_1 | \mathbf{b}_1 \rangle}\mathbf{b}_1 = (27/9)\mathbf{b}_1 = (3,6,6)^T$.

5. On $\mathbb{R}^3$, let $\mathbf{b}_1 = (3,4,5)^T$, $\mathbf{b}_2 = (5,-10,5)^T$, $\mathbf{b}_3 = (7,1,-5)^T$. Note that these vectors are orthogonal. Let $\mathbf{v} = (2,1,2)^T$.

a) Write $\mathbf{v}$ as a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$.

$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$, where $c_1 = \langle \mathbf{b}_1 | \mathbf{v} \rangle / \langle \mathbf{b}_1 | \mathbf{b}_1 \rangle = 20/50 = 2/5$, $c_2 = \langle \mathbf{b}_2 | \mathbf{v} \rangle / \langle \mathbf{b}_2 | \mathbf{b}_2 \rangle = 10/150 = 1/15$, $c_3 = \langle \mathbf{b}_3 | \mathbf{v} \rangle / \langle \mathbf{b}_3 | \mathbf{b}_3 \rangle = 5/75 = 1/15$.

b) Let M be a matrix with eigenvalues $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_3 = 15$ and eigenvectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$. Compute $M\mathbf{v}$.

$M\mathbf{v} = c_1\lambda_1\mathbf{b}_1 + c_2\lambda_2\mathbf{b}_2 + c_3\lambda_3\mathbf{b}_3 = 2\mathbf{b}_1 + (1/5)\mathbf{b}_2 + \mathbf{b}_3 = (14,7,6)^T$.