M346 Second Midterm Exam, October 23, 2009

1) Diagonalization:

a)(10 pts) Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & 8 \\ 2 & -3 \end{pmatrix}$

Since the trace is zero and the determinant is -25, the eigenvalues are ± 5. *The eigenvectors are* $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

b) (10 pts) Compute e^{Bt}, where $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

Since the trace is 5 and the determinant is zero, the eigenvalues are 0 *and* 5. *Computing the eigenvectors we get* $P = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$, $P^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, $e^{Bt} = Pe^{Dt}P^{-1} = \frac{1}{5}\begin{pmatrix} 4 + e^{5t} & -2 + 2e^{5t} \\ -2 + 2e^{5t} & 1 + 4e^{5t} \end{pmatrix}$.

c) (10 pts) Find the eigenvalues of $C = \begin{pmatrix} 2 & 2 & 3 & 7 \\ 2 & 5 & 2 & 8 \\ 0 & 0 & 6 & 16 \\ 0 & 0 & 4 & -6 \end{pmatrix}$. You do not need to find the eigenvectors.

This is block triangular. The upper left block is just $B+I$, *with eigenvalues 1 and 6. The lower right block is* $2A$, *with eigenvalues 10 and -10, so* C *has eigenvalues 1, 6, 10, and -10.*

2. Consider the system of equations $\mathbf{x}(n+1) = A\mathbf{x}(n)$, where $A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$, with initial condition $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

a) (15 pts) Find $\mathbf{x}(n)$ for all n. Be as explicit as possible.

The eigenvalues of A *are 1 and* -2, *with eigenvectors* $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and* $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. *Since* $\mathbf{x}(0) = \mathbf{b}_1 - 2\mathbf{b}_2$, $\mathbf{x}(n) = 1^n\mathbf{b}_1 + (-2)^{n+1}\mathbf{b}_2 = \begin{pmatrix} 1 + (-2)^{n+1} \\ 1 + (-2)^{n+2} \end{pmatrix}$.

b) (5 pts) Find $\lim_{n\to\infty} \frac{x_1(n)}{x_2(n)}$.

You don't need to complete (a) to get this. In the long run, $\mathbf{x}$ *points in the direction of the dominant eigenvector* $\mathbf{b}_2$, *so* x_1/x_2 *approaches* $-1/2$.

c) (5 pts) Find $\lim_{n\to\infty} \frac{x_1(n+1)}{x_1(n)}$. *Likewise,* $\mathbf{x}$ *gets multiplied by the dominant eigenvalue each turn, so* $x_1(n+1)/x_1(n)$ *approaches* -2.

3. a) (15 pts) Consider the system of nonlinear coupled differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(3 - 2x_1 - x_2)\\ \frac{dx_2}{dt} &= x_2(3 - x_1 - 2x_2).\end{aligned}$$

This system of equations has four fixed points, namely $(0,0)^T$, $(3/2,0)^T$, $(0,3/2)^T$, and $(1,1)^T$. These equations describe competition between two species in the same environment. For each fixed point, find the matrix that describes the linearization near the fixed point, indicate how many stable, neutral, and unstable modes there are, and indicate whether the fixed point is stable, unstable, or neutral.

Since $f_1(x_1,x_2) = 3x_1 - 2x_1^2 - x_1x_2$ *and* $f_2(x_1,x_2) = 3x_2 - x_1x_2 - 2x_2^2$, *our matrix of partial derivatives is* $A = \begin{pmatrix} 3-4x_1-x_2 & -x_1 \\ -x_2 & 3-4x_2-x_1 \end{pmatrix}$.

Around (0,0), we have $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$*, which has two unstable modes, so the system is unstable.*

Around (3/2,0), $A = \begin{pmatrix} -3 & -3/2 \\ 0 & 3/2 \end{pmatrix}$ *with eigenvalues -3 and 3/2, so there is one stable and one unstable mode, and the system is unstable.*

Around (0,3/2), $A = \begin{pmatrix} 3/2 & 0 \\ -3/2 & -3 \end{pmatrix}$ *with eigenvalues -3 and 3/2, so there is one stable and one unstable mode, and the system is unstable.*

Around (1,1), $A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$*, with eigenvalues -3 and -1. Both modes are stable, so the system is stable. In the long run, the populations (in appropriate units) of both species will approach 1.*

3 b) (10 pts) Now consider the equations

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(3 - x_1 - 2x_2)\\ \frac{dx_2}{dt} &= x_2(3 - 2x_1 - x_2).\end{aligned}$$

These equations describe somewhat more intense competition, and have fixed points $(0,0)^T$, $(3,0)^T$, $(0,3)^T$ and $(1,1)^T$. For each fixed point, find the matrix that describes the linearization near the fixed point, indicate how many stable, neutral, and unstable modes there are, and indicate whether the fixed point is stable, unstable, or neutral.

Since $f_1(x_1,x_2) = 3x_1 - x_1^2 - 2x_1x_2$ *and* $f_2(x_1,x_2) = 3x_2 - 2x_1x_2 - x_2^2$, *our matrix of partial derivatives is* $A = \begin{pmatrix} 3-2x_1-2x_2 & -2x_1 \\ -2x_2 & 3-2x_2-2x_1 \end{pmatrix}$.

Around (0,0), we have $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, *which has two unstable modes, so the system is unstable.*

Around (3,0), $A = \begin{pmatrix} -3 & -6 \\ 0 & -3 \end{pmatrix}$ *with eigenvalues -3 and -3, so there are two stable modes. If species one starts off outnumbering species two, the system will approach this fixed points, which represents the extinction of species two.*

Around (0,3), $A = \begin{pmatrix} -3 & 0 \\ -3 & -3 \end{pmatrix}$ *with eigenvalues -3 and -3, with everything stable. If species two starts off outnumbering species one, the system will approach this fixed points, which represents the extinction of species one.*

Around (1,1), $A = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$, *with eigenvalues -3 and +1. One mode is stable and the other is unstable, with the whole system being unstable. Any imbalance between the populations will eventually lead to the extinction of one of the two species.*

Problem 3a described mild competition, reaching an equilibrium where both species survive. Problem 3b describes killer competition, literally.

4. True of False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given. For all questions, A is a square real matrix.

a) The eigenvalues of A must be real.

False. A real matrix can have complex eigenvalues.

b) If the trace of a A is positive, then the system of equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ is unstable.

True. If the trace is positive, then at least one eigenvalue has a positive real part.

c) If the trace of a A is negative, then the system of equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ is stable.

False. There is at least one stable mode, but there may also be unstable *modes.*

d) The geometric multiplicity of an eigenvalue can be greater than the algebraic multiplicity.

False. The geometric multiplicity is less than or equal to the algebraic multiplicity.

e) If A is a 5×5 matrix whose characteristic polynomial has 5 distinct roots, then A is diagonalizable.

True. Five roots means five eigenvectors, which form a basis.