M346 Third Midterm Exam Solutions, November 20, 2009

1) Gram Schmidt:

a)(10 points) On $\mathbb{R}^3$ with the usual inner product, Use Gram-Schmidt to convert $\mathbf{x}_1 = (1,2,0)^T$, $\mathbf{x}_2 = (3,1,1)^T$, $\mathbf{x}_3 = (4,3,-5)^T$ to an orthogonal basis.

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{y}_1 | \mathbf{x}_2 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle} \mathbf{y}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{y}_1 | \mathbf{x}_3 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle} \mathbf{y}_1 - \frac{\langle \mathbf{y}_2 | \mathbf{x}_3 \rangle}{\langle \mathbf{y}_2 | \mathbf{y}_2 \rangle} \mathbf{y}_2 = \begin{pmatrix} 4 \\ 3 \\ -5 \end{pmatrix} - \frac{10}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{0}{6} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix}$$

b)(15 points) On $\mathbb{R}_2[t]$ with the inner product $\langle f|g \rangle = \int_0^2 f(t)g(t)dt$, transform $\{1, t, t^2\}$ to an orthogonal basis.

$$\mathbf{y}_1 = \mathbf{x}_1 = 1$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\int_0^2 t\, dt}{\int_0^2 1\, dt} \mathbf{y}_1 = t - 1$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\int_0^2 t^2\, dt}{\int_0^2 1\, dt} \mathbf{y}_1 - \frac{\int_0^2 t^2(t-1)\, dt}{\int_0^2 (t-1)^2\, dt} \mathbf{y}_2 = t^2 - \frac{4}{3} - 2(t-1) = t^2 - 2t + \frac{2}{3}$$

2. a)(15 points) Find the equation of the best line through the points $(1,-4)$, $(2,1)$, and $(3,2)$.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$, $A^T A = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$, $A^T \mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, and $\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = (A^T A)^{-1}(A^T \mathbf{b}) = \begin{pmatrix} -19/3 \\ 3 \end{pmatrix}$, so the best line is $y = 3x - 19/3$.

b)(10 points) Let $V$ be the subspace of $\mathbb{R}^3$ that is the span of the vectors $(1,2,3)^T$ and $(1,1,1)^T$. Find the point in $V$ that is closest to $(-4,1,2)^T$.

This is essentially the same problem, since the least-squares solution to $A\mathbf{x} = \mathbf{b}$ places $A\mathbf{x}$ as close as possible to $\mathbf{b}$. Our answer is $A\begin{pmatrix} -19/3 \\ 3 \end{pmatrix} = \begin{pmatrix} -10/3 \\ -1/3 \\ 8/3 \end{pmatrix}$.

3. On $\mathbb{C}^3$ with the usual inner product, let

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 - ix_3 \\ 2x_2 + (1-i)x_3 \\ ix_1 + 3x_2 + x_3 \end{pmatrix}$$

a)(5 points) Find the matrix of $L$:

$L = \begin{pmatrix} 1 & i & -i \\ 0 & 2 & 1-i \\ i & 3 & 1 \end{pmatrix}$

b)(10 points) Let $\mathbf{x} = \begin{pmatrix} 1 \\ 10 \\ 100 \end{pmatrix}$. Compute $L^\dagger(\mathbf{x})$. Since $L^\dagger = \begin{pmatrix} 1 & 0 & -i \\ -i & 2 & 3 \\ 1 & i+1 & 1 \end{pmatrix}$,

$L^\dagger \mathbf{x} = \begin{pmatrix} 1-100i \\ 320-i \\ 110+11i \end{pmatrix}$.

c)(10 points) Let $V$ be the space of real-valued functions on the real line, with the inner product $\langle f|g\rangle = \int_{-\infty}^{\infty} f(t)g(t)dt$. Let $A : V \to V$ be the linear transformation $A = t + d/dt$ (That is, $(A(f))(t) = tf(t) + f'(t)$). Let $g(t) = e^{-t^2/2}$. Compute $Ag$ and $A^\dagger g$.

We saw in class that the adjoint to $d/dt$ is $-d/dt$, while multiplication by $t$ is self-adjoint, so $A^\dagger = t - d/dt$. It's then an easy calculation to get $Ag = 0$, $A^\dagger g(t) = 2te^{-t^2/2}$. [Physics note: In quantum mechanics, $g$ is the wave function of the ground state of a harmonic oscillator. The operators $A$ and $A^\dagger$ are called "ladder operators", or "raising and lowering operators". $A^\dagger$ increases the energy level by one, and $2te^{-t^2/2}$ is the wave function for the first excited state. $A$ lowers the energy by one. Since there's nothing below the ground state, we have $Ag = 0$.]

4. Grab bag. These are short-answer or true/false questions. Each question is worth 5 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) True or false? The matrix $\begin{pmatrix} 5 & 4i \\ -4i & -1 \end{pmatrix}$ has orthogonal eigenvectors.

True. The matrix is Hermitian.

b) True or false? The matrix $\frac{1}{\sqrt{7}} \begin{pmatrix} 2-i & -1+i \\ 1+i & 2+i \end{pmatrix}$ is unitary.

True. The columns are orthonormal.

c) Let $\mathbf{x}(t)$ be the solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, where $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & -1 & 4 \\ -2 & 1 & 0 & 5 \\ -3 & -4 & -5 & 0 \end{pmatrix}$ and $\mathbf{x}(0) = (5, -3, 1, 1)^T$ Find the limit, as $t \to \infty$, of $|\mathbf{x}(t)|$. (This has a quick and easy solution, and you do NOT have to diagonalize $A$!)

Since $A$ is anti-symmetric, $e^{At}$ is orthogonal, so $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$ has the same length as $\mathbf{x}$, namely 6.

d) True or false? If a matrix $M$ satisfies $M = M^T$, then the eigenvalues of $M$ are real.

False. Some of the matrix elements of $M$ may be complex, in which case $M$ won't be Hermitian. (E.g., $M$ could be $i$ times the identity)

e) True or false? If a matrix is unitary, then it is not Hermitian.

False. The identity matrix is both Hermitian and unitary. More generally, any diagonalizable matrix with orthogonal eigenvectors and who eigenvalues are 1 and -1 is both Hermitian and unitary.