M346 Second Midterm Exam, March 27, 2009

1a) Find a matrix with eigenvalues 3 and 4 and eigenvectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, respectively.

b) Find a matrix with eigenvalues $3 \pm 2i$ and eigenvectors $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$.

2. Find the eigenvalues of $\begin{pmatrix} 3 & 5 & 7 & 6 \\ 0 & 3 & 1 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 4 & 3 & 15 \end{pmatrix}$. You do not have to find the eigenvectors.

3. The matrix $A = \begin{pmatrix} 3.3 & -2.1 \\ -2.1 & -2.3 \end{pmatrix}$ has eigenvalues $-3$ and $4$, with eigenvectors $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$. We wish to solve $\mathbf{x}(n+1) = A\mathbf{x}(n)$. As usual, let $\mathbf{y} = [\mathbf{x}]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis of eigenvectors.

a) If $\mathbf{x}(0) = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$, what is $\mathbf{y}(0)$?

b) Find $\mathbf{y}(n)$ for all $n \geq 0$.

c) Find $\mathbf{x}(n)$ for all $n \geq 0$.

4. a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 3 \\ 2 & 8 \end{pmatrix}$. Double-check that your eigenvectors are correct, as you will need them for the other parts!

b) If $d\mathbf{x}/dt = A\mathbf{x}$ and $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, what is the limit, as $t \to \infty$, of $x_1(t)/x_2(t)$? [Note: you do not have to actually compute $\mathbf{x}(t)$ to do this!]

Extra credit. If $\mathbf{x}(n+1) = A\mathbf{x}(n)$ and $\mathbf{x}(0) = \begin{pmatrix} 3.14159 \\ 2.71828 \end{pmatrix}$, what is the limit, as $n \to \infty$, of $x_1(n+1)/x_1(n)$? [For heaven's sake, don't attempt to do arithmetic with these numbers! Think about the long-term behavior we discussed on Wednesday.]