M346 Second Midterm Exam, March 27, 2009

1a) Find a matrix with eigenvalues 3 and 4 and eigenvectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, respectively.

$A = PDP^{-1}$, where $P = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$. Since the determinant of P is 1, we compute $P^{-1} = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$ and multiply things out to get $A = \begin{pmatrix} -5 & 12 \\ -6 & 12 \end{pmatrix}$.

b) Find a matrix with eigenvalues $3 \pm 2i$ and eigenvectors $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$.

$A = PDP^{-1}$, where $P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ and $D = \begin{pmatrix} 3+2i & 0 \\ 0 & 3-2i \end{pmatrix}$. Since the determinant of P is -2i, we compute $P^{-1} = \frac{i}{2}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$ and multiply things out to get $A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$.

2. Find the eigenvalues of $\begin{pmatrix} 3 & 5 & 7 & 6 \\ 0 & 3 & 1 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 4 & 3 & 15 \end{pmatrix}$. You do not have to find the eigenvectors.

The matrix is block-triangular, with a northwest 1×1 block and a southeast 3×3 block. The 3×3 block is itself block triangular. So we're left with 3, 15, and the eigenvalues of $\begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}$. The rows add up to 4, and the trace is 2, so the eigenvalues of the 2×2 block must be 4 and -2. The eigenvalues of the whole matrix are $\{3, 4, -2, 15\}$.

3. The matrix $A = \begin{pmatrix} 3.3 & -2.1 \\ -2.1 & -2.3 \end{pmatrix}$ has eigenvalues -3 and 4, with eigenvectors $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$. We wish to solve $\mathbf{x}(n+1) = A\mathbf{x}(n)$. As usual, let $\mathbf{y} = [\mathbf{x}]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis of eigenvectors.

a) If $\mathbf{x}(0) = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$, what is $\mathbf{y}(0)$?

$P = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$, $P^{-1} = \frac{1}{10}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$, and $\mathbf{y}(0) = P^{-1}\mathbf{x}(0) = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$. You should check that $\mathbf{x}(0) = 7\mathbf{b}_1 - \mathbf{b}_2$.

b) Find $\mathbf{y}(n)$ for all $n \geq 0$.

Just multiply each coefficient by λ^n. $\mathbf{y}(n) = \begin{pmatrix} 7 \cdot (-3)^n \\ -1 \cdot 4^n \end{pmatrix}$.

c) Find $\mathbf{x}(n)$ for all $n \geq 0$.

$\mathbf{x}(n) = P\mathbf{y}(n) = \begin{pmatrix} 7 \cdot (-3)^n + 3 \cdot 4^n \\ 21 \cdot (-3)^n - 4^n \end{pmatrix}$.

4. a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 3 \\ 2 & 8 \end{pmatrix}$. Double-check that your eigenvectors are correct, as you will need them for the other parts!

The trace is 11 and the determinant is 18, so the eigenvalues are 9 and 2, with eigenvectors (obtained by row reduction) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

b) If $d\mathbf{x}/dt = A\mathbf{x}$ and $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, what is the limit, as $t \to \infty$, of $x_1(t)/x_2(t)$? [Note: you do not have to actually compute $\mathbf{x}(t)$ to do this!]

Since $\mathbf{x}(0)$ is not an eigenvector, it is of the form $c_1\mathbf{b}_1 + c_2\mathbf{b}_2$, with c_1 and c_2 nonzero. Then $\mathbf{x}(t) = c_1 e^{9t}\mathbf{b}_1 + c_2 e^{2t}\mathbf{b}_2$, which asymptotically points in the $\mathbf{b}_1$ direction. So our limit is 1/2.

Extra credit. If $\mathbf{x}(n+1) = A\mathbf{x}(n)$ and $\mathbf{x}(0) = \begin{pmatrix} 3.14159 \\ 2.71828 \end{pmatrix}$, what is the limit, as $n \to \infty$, of $x_1(n+1)/x_1(n)$? [For heaven's sake, don't attempt to do arithmetic with these numbers! Think about the long-term behavior we discussed on Wednesday.]

In this example, $\mathbf{x}(n) = c_1 9^n \mathbf{b}_1 + c_2 2^n \mathbf{b}_2$ for some ugly numbers c_1 and c_2. The first term dominates, and it just gets multiplyied by 9 each time, so our limit is 9.