M346 Second Midterm Exam, August 5, 2011

1) (25 pts) Consider the matrix $A = \begin{pmatrix} 4 & 2 \\ 8 & -2 \end{pmatrix}$.

a) Find the eigenvalues of A. For each eigenvalue, find a corresponding eigenvector.

b) Compute A^n for all n. Make your answer as explicit as possible.

c) Compute e^{At} as a function of t. Make your answer as explicit as possible.

2. (30 points, 2 pages) Let $A = \begin{pmatrix} -4 & 5 \\ 5 & -4 \end{pmatrix}$. This matrix has eigenvalues 1 and -9 and corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a) Find the solution to $\mathbf{x}(n+1) = A\mathbf{x}(n)$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

b) Find the solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

c) Find the solution to $\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}$ with initial conditions $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\dot{\mathbf{x}}(0) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

3. (15 pts) The differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= e^{-4x_1} - x_2 \\ \frac{dx_2}{dt} &= 5x_1x_2 \end{aligned}$$

have a fixed point at $x_1 = 0$, $x_2 = 1$. Find the linear approximation to these equations near $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and determine whether this fixed point is stable. Explain your reasoning!

4. (15 points)

On $\mathbb{R}_2[t]$ with the inner product $\langle \mathbf{f}|\mathbf{g}\rangle = \int_0^1 f(t)g(t)dt$, use Gram-Schmidt to convert $\{1, 2t, 6t^2\}$ to an orthogonal basis.

5. (15 points) In $\mathbb{R}^5$ with the standard inner product, let V be the subspace spanned by $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \\ 5 \\ -2 \\ 1 \end{pmatrix}$. Write $\mathbf{b} = \begin{pmatrix} 7 \\ 4 \\ 5 \\ 1 \\ 2 \end{pmatrix}$ as the sum of two vectors, one in V and the other orthogonal to V.