M346 Final Exam Solutions, May 14, 2011

1) In $\mathbb{R}^2$, *consider the operator* $L(\mathbf{x}) = A\mathbf{x}$, *where* $A = \begin{pmatrix} 5 & 10 \\ -15 & 20 \end{pmatrix}$. *Consider the basis* $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ *and the vector* $\mathbf{x} = \begin{pmatrix} 120 \\ 70 \end{pmatrix}$.

a) Find the coordinates of $\mathbf{x}$ *in the* $\mathcal{B}$ *basis. (That is, find* $[\mathbf{x}]_{\mathcal{B}}$*.)*

$P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, so $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$, so $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}\mathbf{x} = \begin{pmatrix} 58 \\ 4 \end{pmatrix}$. You can check that $\begin{pmatrix} 120 \\ 70 \end{pmatrix}$ is indeed $58\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

b)Find the coordinates of L *in the* $\mathcal{B}$ *basis, that is* $[L]_{\mathcal{B}}$.

$[L]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}} A P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 14 & 12 \\ -8 & 11 \end{pmatrix}$.

2. Let $V = \mathbb{R}_2[t]$, *the space of quadratic polynomials in a variable* t. *On* V, *consider the operator* $(L(\mathbf{p}))(t) = \mathbf{p}(2t+1)$, *where the right hand side means the polynomial* $\mathbf{p}$ *evaluated at the point* $2t+1$. *(If* $\mathbf{p}(t)$ *were the function sin*(t), *then* $L(\mathbf{p})$ *would be the function* $\sin(2t+1)$. *Of course,* $\mathbf{p}$ *is a polynomial rather than a trig function, but the rule for how* L *acts is the same.)*

a) Find the matrix of L *with respect to the basis* $\mathcal{E} = \{1, t, t^2\}$.

$L(1) = 1$, $L(t) = 1 + 2t$, and $L(t^2) = (2t+1)^2 = 1 + 4t + 4t^2$, so the matrix is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}$.

b) Find all solutions to $L(\mathbf{p}) = 2\mathbf{p}$.

This is another way of saying "find all eigenvectors of L with eigenvalue 2". These are the vectors in V whose coordinates are eigenvectors of $[L]_{\mathcal{B}}$ with eigenvalue 2. By row reduction, you get that $[\mathbf{p}]_{\mathcal{B}}$ must be a multiple of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, so $\mathbf{p}$ itself must be a multiple of $1 + t$.

3. Diagonalization.

a) Find the characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & -2 & -3 \\ -3 & 0 & -2 \\ -4 & -1 & 0 \end{pmatrix}$.

You do not need to find the eigenvalues or eigenvectors.

$p_A(\lambda) = \det \begin{pmatrix} \lambda - 1 & 2 & 3 \\ 3 & \lambda & 2 \\ 4 & 1 & \lambda \end{pmatrix} = \lambda^3 - \lambda^2 - 20\lambda + 27$. No, I don't expect you to find the roots of this!

Note that the definition of $p_A(\lambda)$ is $\det(\lambda I - A)$, not $\det(A - \lambda I)$! These expressions differ by a factor of $(-1)^n$.

b) Find the eigenvalues of $B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -3 & 6 & 0 & 0 \\ 3 & 5 & 2 & 3 \\ 2 & 9 & -3 & 2 \end{pmatrix}$. *You do not need to find the eigenvectors.*

This is block triangular, so we just need to find the eigenvalues of $\begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix}$ and the eigenvalues of $\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$. These can be done by finding the characteristic polynomial and finding its roots, but there are easier ways. For the first matrix, the rows all sum to 3 and the trace is 8, so the eigenvalues must be 3 and 5. For the second matrix, this is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with eigenvalues $a \pm bi$, so our eigenvalues oare $2 - 3i$ and $2 + 3i$. Put everything together, and the eigenvalues are 3, 5, and $2 \pm 3i$.

c) Find the eigenvalues and eigenvectors of $C = \begin{pmatrix} 5 & 2 \\ -1 & 2 \end{pmatrix}$.

The trace is 7 and the determinant is 12, so the eigenvalues are 3 and 4. (Or you can compute the characteristic polynomial and apply the quadratic formula.) By row reduction, the eigenvectors are found to be $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

4. Consider the matrix $A = \frac{1}{5} \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$.

a) Is the system of equations $\mathbf{x}(n+1) = A\mathbf{x}(n)$ *stable or unstable? What is/are the dominant eigenvalue(s)?*

The eigenvalues of A are $-4/5$ and $(1 \pm 2i)/5$, with corresponding eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{b}_{2,3} = \begin{pmatrix} 0 \\ \pm i \\ 1 \end{pmatrix}$. All of these have magnitude less than 1, so the system is stable. The biggest eigenvalue (in magnitude) is $-4/5$, so that's the dominant eigenvalue.

b) Find a solution to $\mathbf{x}(n+1) = A\mathbf{x}(n)$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(You can leave your answer as a linear combination of eigenvectors.)

$\mathbf{x}(0) = \mathbf{b}_1 - \frac{i}{2}\mathbf{b}_2 + \frac{i}{2}\mathbf{b}_3$, so $\mathbf{x}(n) = (-4/5)^n \mathbf{b}_1 - \frac{i}{2}\left(\frac{1+2i}{5}\right)^n \mathbf{b}_2 + \frac{i}{2}\left(\frac{1-2i}{5}\right)^n \mathbf{b}_3$.

c) Now consider the system of differential equations $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. Is the system stable, neutral, or unstable? What is/are the dominant eigenvalues?

Since $(1 \pm 2i)/5$ have positive real part, the system is unstable. The dominant eigenvalue is the one with the greatest real part. This is a tie between $(1 \pm 2i)/5$.

d) Find a solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$\mathbf{x}(t) = e^{-4t/5}\mathbf{b}_1 - \frac{i}{2}e^{\frac{(1+2i)t}{5}}\mathbf{b}_2 + \frac{i}{2}e^{\frac{(1-2i)t}{5}}\mathbf{b}_3$. This has the same form as the solution to part (b), only with $e^{\lambda t}$ instead of λ^n. If you add up the different terms, it actually simplifies greatly, to $\mathbf{x}(t) = \begin{pmatrix} e^{-4t/5} \\ e^t \cos(2t) \\ e^t \sin(2t) \end{pmatrix}$.

5. Orthogonality. In $\mathbb{R}^3$, let V be the span of the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$.

This problem is essentialy Gram-Schmidt applied to the three vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 70 \\ 0 \\ 0 \end{pmatrix}$, except that I asked about the first two vectors in part (a) and the third vector in part (b).

a) Use Gram-Schmidt to find an orthogonal basis for V.

$\mathbf{y}_1 = \mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{y}_1|\mathbf{x}_2\rangle}{\langle \mathbf{y}_1|\mathbf{y}_1\rangle}\mathbf{y}_1 = \mathbf{x}_2 - \frac{28}{14}\mathbf{y}_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

b) Let $\mathbf{x} = \begin{pmatrix} 70 \\ 0 \\ 0 \end{pmatrix}$. *Write* $\mathbf{x}$ *as the sum of two vectors, one in* V *and the other orthogonal to* V.

We want to write $\mathbf{x} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} \in V$ and $\mathbf{w} \perp V$. $\mathbf{v}$ is the projection of $\mathbf{x}$ onto V, and equals $\frac{70}{14}\mathbf{y}_1 + \frac{-210}{10}\mathbf{y}_2 = \begin{pmatrix} 68 \\ 10 \\ -6 \end{pmatrix}$. $\mathbf{w}$ is what's left over, namely $\begin{pmatrix} 2 \\ -10 \\ 6 \end{pmatrix}$.

6. a) On $\mathbb{C}^3$, *let the operator* L *be given by the rule* $L(\mathbf{x}) = \begin{pmatrix} 3x_1 + 5x_2 - x_3 \\ 4x_1 + ix_2 + x_3 \\ 7x_1 - x_2 + ix_3 \end{pmatrix}$. *Compute* $L^\dagger(\mathbf{x})$.

The matrix of L is $\begin{pmatrix} 3 & 5 & -1 \\ 4 & i & 1 \\ 7 & -1 & i \end{pmatrix}$, so the matrix of $L^\dagger$ is $\begin{pmatrix} 3 & 4 & 7 \\ 5 & -i & -1 \\ -1 & 1 & -i \end{pmatrix}$, so $L^\dagger(\mathbf{x}) = \begin{pmatrix} 3x_1 + 4x_2 + 7x_3 \\ 5x_1 - ix_2 - x_3 \\ -x_1 + x_2 - ix_3 \end{pmatrix}$.

b) Let $A = \begin{pmatrix} 0 & -3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$, *let* $B = e^A$, *and let* $C = e^{\pi A}$. *Which of these matrices are Hermitian? Which are anti-hermitian? Which are orthogonal? Explain.*

A is real anti-symmetric, hence anti-hermitian, which means that the eigenvalues are pure imaginary ($\pm 3i$ and $\pm 2i$) and the eigenvectors are orthogonal. The eigenvectors of A are also the eigenvectors of B and C. The eigenvalues of B are $e^{\pm 2i}, e^{\pm 3i}$ which are complex and on the unit circle, while the eigenvalues of C are $e^{\pm 2\pi i} = 1$ and $e^{\pm 3\pi i} = -1$. These are both on the unit circle and are real. Since A is a real matrix, both B and C are real matrices. Thus, C is Hermitian, A is anti-Hermitian, and both B and C are orthogonal.

7. Working on the interval $[0,1]$*, let* $f_0(x) = 1$ *for* $0 < x < 1$*. We write this function as a Fourier series* $f_0(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$*.*

a) Compute the coefficients a_n*.*

$a_n = \frac{2}{1}\int_0^1 1 \times \sin(n\pi x)dx = \frac{-2}{n\pi}\cos(n\pi x)|_0^1$. This equals $\frac{4}{n\pi}$ if n is odd and 0 if n is even.

b) Now suppose that $f(x,t)$ *satisfies the "heat equation"*

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2},$$

with Dirichlet boundary conditions $f(0,t) = f(1,t) = 0$*. [Physical note:* $f(x,t)$ *describes the temperature of a point* x *along a rod of length 1 at time* t*, where the ends of the rod are in contact with heat sinks at temperature 0.] Viewing this as an ordinary differential equation (*$d\mathbf{f}/dt = L(\mathbf{f})$*) in a space of functions, what is the dominant mode? Is it stable or unstable? How quickly does it grow or shrink?*

This is analogous to solving the wave equation in terms of standing waves, except that our equation is $\frac{d\mathbf{f}}{dt} = A\mathbf{f}$ rather than $\frac{d^2\mathbf{f}}{dt^2} = A\mathbf{f}$, where A is the operator d^2/dx^2. As with the standing wave calculation, the eigenvalues of A are $-n^2\pi^2$, and the eigenvectors are $\sin(n\pi x)$. All eigenvalues are negative, meaning that all terms shrink away with time, but the least negative eigenvalue (i.e. the dominant eigenvalue) is $-\pi^2$, meaning that the term that survives the longest goes as $e^{-\pi^2 t}$.

c) Find the solution $f(x,t)$ *for all (non-negative)* t*, starting with initial condition* $f(x,0) = f_0(x)$*. You can leave your answer as a series. [Note: The initial condition is discontinuous at* $x = 0$ *and* $x = 1$*, since the entire rod is hot but the surroundings are cold, but the solution quickly becomes smooth.]*

$f(x,t) = \sum a_n e^{\lambda_n t}\sin(n\pi x) = \sum_{n \text{ odd}} \frac{4}{n\pi} e^{-n^2\pi^2 t}\sin(n\pi x)$.