M346 First Midterm Exam Solutions, February 17, 2011

1) (15 points) Consider the vectors $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ in $\mathbb{R}^3$. Are these vectors linearly independent? Do they span $\mathbb{R}^3$? Do they form a basis for $\mathbb{R}^3$?

Answer: Row-reducing the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 5 & 5 \end{pmatrix}$, whose columns are the vectors in question, yields $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$. There are only two pivots, so the vectors are not linearly independent (since there are more than two columns), they do not span (since there are more than two rows), and they do not form a basis.

2. (15 points) Let $V = \mathbb{R}_2[t]$ be the space of quadratic polynomials in a variable t and consider the linear transformation $L(\mathbf{p}) = (t+1)\mathbf{p}'(t)$ from V to itself, where $\mathbf{p}'(t)$ is the derivative of $\mathbf{p}(t)$. Find the matrix of this linear transformation with respect to the (standard) basis $\{1, t, t^2\}$.

Answer: Since $L(1) = 0$, $L(t) = t + 1$ and $L(t^2) = 2t(t+1) = 2t^2 + 2t$, our matrix is $(\,[L(1)]_{\mathcal{E}}[L(t)]_{\mathcal{E}}[L(t^2)]_{\mathcal{E}}\,) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 1 & -1 & -1 & 1 & 8 \\ 1 & 2 & 8 & 3 & 7 \\ 1 & 2 & 8 & -2 & -28 \\ 1 & 5 & 17 & 0 & -29 \end{pmatrix}$. A is row-equivalent to $\begin{pmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & 3 & 0 & -5 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

a) Find a basis for the null space of A.

Answer: The equations $A_{rref}\mathbf{x} = 0$ are essentially:

$$\begin{array}{rrr} x_1 = & -2x_3 & +4x_5 \\ x_2 = & -3x_3 & +5x_5 \\ x_3 = & x_3 & \\ x_4 = & & -7x_5 \\ x_5 = & & x_5, \end{array}$$

so our basis is $\left\{ \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 0 \\ -7 \\ 1 \end{pmatrix} \right\}$.

b) Find a basis for the column space of A.

Answer: Since there are pivots in the first, second and fourth columns, we want the first, second and fourth columns of A, namely $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \end{pmatrix} \right\}$

4. a) In $\mathbb{R}^2$, let $\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \end{pmatrix} \right\}$ be a basis, and let $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Let $\mathcal{E}$ be the standard basis. Compute the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$ and compute the coordinates of $\mathbf{x}$ in the $\mathcal{B}$ basis.

Answer: $P_{\mathcal{E}\mathcal{B}} = (\mathbf{b}_1\mathbf{b}_2) = \begin{pmatrix} 3 & 5 \\ 5 & 8 \end{pmatrix}$. $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} -8 & 5 \\ 5 & -3 \end{pmatrix}$. (Remember the formula for the inverse of a 2×2 matrix!) Then $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\mathbf{x}]_{\mathcal{E}} = \begin{pmatrix} -8 & 5 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -18 \\ 11 \end{pmatrix}$. We can check that we did things right by computing $-18\mathbf{b}_1 + 11\mathbf{b}_2$ and seeing that it really is $\mathbf{x}$.

b) In $\mathbb{R}_1[t]$, let $\mathcal{D} = \{3 + 5t, 5 + 8t\}$ and let $\mathbf{p}(t) = 1 + t$. Compute $[\mathbf{p}]_{\mathcal{D}}$.

Answer: This is similar, as $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$ are the same as in part (a). Since $[\mathbf{p}]_{\mathcal{E}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $[\mathbf{p}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. As a check, $-3(3 + 5t) + 2(5 + 8t)$ does equal $1 + t$.

5. a) Find the characteristic polynomial of $\begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix}$. (You do not have to compute the eigenvalues or eigenvectors).

Answer: $p_A(\lambda) = det \begin{pmatrix} \lambda - 3 & -2 \\ -5 & \lambda - 1 \end{pmatrix} = (\lambda - 3)(\lambda - 1) - 10 = \lambda^2 - 4\lambda - 7.$

b) Find the eigenvalues of $\begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$. (You do not have to find the eigenvectors).

Answer: In this case, $p_A(\lambda) = (\lambda - 1)^2 + 4 = \lambda^2 - 2\lambda + 5$. From the quadratic formula, or by completing the square (which was complete to begin with), the roots are $1 \pm 2i$.

c) $\lambda = 2$ is one of the eigenvalues of $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$. Find a basis for the corresponding eigenspace.

Answer: $2I - A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$ row-reduces to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, hence the equations $x_1 = -x_2 - x_3$, $x_2 = x_2$, $x_3 = x_3$, and basis $\{(-1,1,0)^T, (-1,0,1)^T\}$ for E_2.