M346 Second Midterm Exam Solutions, April 7, 2011

1) The matrix $A = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -4$, with eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Suppose that $\mathbf{x}(n)$ satisfies the system of equations $\mathbf{x}(n+1) = A\mathbf{x}(n)$ for all $n \geq 0$.

a) If $\mathbf{x}(0)$ is "random" (meaning any nonzero vector that isn't an eigenvector of A), compute the limits $\lim_{n\to\infty} \frac{x_1(n)}{x_2(n)}$ and $\lim_{n\to\infty} \frac{x_1(n+1)}{x_1(n)}$. In other words, what is the asymptotic direction of $\mathbf{x}(n)$ and the asymptotic growth rate?

Since $|-4| > |1|$, the dominant eigenvalue is λ_2, with dominant eigenvector $\mathbf{b}_2$. Asymptotically, $\mathbf{x}$ will point in the $\mathbf{b}_2$ direction and grow by a factor of -4 each turn, so the two answers are $-1/4$ and -4, respectively.

b) Now suppose that $\mathbf{x}(0) = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$. Find $\mathbf{x}(n)$ exactly for all n.

Since $\mathbf{x}(0) = 3\mathbf{b}_1 + 2\mathbf{b}_2$ (which you can get from change-of-basis matrices, or from row reduction), $\mathbf{x}(n) = 3(1)^n\mathbf{b}_1 + 2(-4)^n\mathbf{b}_2 = \begin{pmatrix} 3 + 2(-4)^n \\ 3 - 8(-4)^n \end{pmatrix}$.

2) Let $A = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$, exactly as in problem 1. Suppose that $\mathbf{x}(t)$ satisfies the differential equation $d\mathbf{x}/dt = A\mathbf{x}$.

a) How many stable and how many unstable modes does this system have? What is the dominant eigenvalue, and what is the dominant eigenvector? For typical initial conditions, compute $\lim_{t\to\infty} x_1(t)/x_2(t)$.

Since $1 > -4$, the dominant eigenvalue is $\lambda = 1$ and the dominant eigenvector is $(1,1)^T$, so the limiting ratio is $1/1 = 1$.

b) Solve the differential equations with initial conditions $\mathbf{x}(0) = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

Since $\mathbf{x}(0) = 3\mathbf{b}_1 + 2\mathbf{b}_2$, $\mathbf{x}(t) = 3e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

3) Let $A = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$, exactly as in problems 1 and 2, only now consider the second order differential equations $d^2\mathbf{x}/dt^2 = A\mathbf{x}$. Write down the most general solution to these equations. (Leave your answer in terms of arbitrary constants, not in terms of initial conditions. I'm not giving you the initial conditions.)

There is one unstable mode with $\lambda_1 > 0$, so $\kappa_1 = \sqrt{1} = 1$, and one (neutrally) stable mode with $\lambda_2 = -4 < 0$, so $\omega_2 = \sqrt{4} = 2$. Our general solution is

$$\mathbf{x}(t) = c_1 \cosh(t)\mathbf{b}_1 + c_2 \sinh(t)\mathbf{b}_1 + c_3 \cos(2t)\mathbf{b}_2 + c_4 \sin(2t)\mathbf{b}_2.$$

This could also be expressed in terms of exponentials.

b) Find a solution corresponding to the initial conditions $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $d\mathbf{x}/dt(0) = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Our constants are $c_1 = y_1(0) = 1$, $c_2 = \dot{y}_1(0)/\kappa_1 = 0$, $c_3 = y_2(0) = 0$, and $c_4 = \dot{y}_2(0)/\omega_2 = 1/2$, in other words

$$\mathbf{x}(t) = \cosh(t)\mathbf{b}_1 + \frac{1}{2}\sin(2t)\mathbf{b}_2$$

4. Consider the nonlinear system of differential equations

$$\frac{dx_1}{dt} = x_1(3 - x_1 - 2x_2); \qquad \frac{dx_2}{dt} = x_2(3 - 2x_1 - x_2)$$

These equations describe the fierce competition between two species for similar resources, where $x_1(t)$ and $x_2(t)$ are the populations of the two species at time t. The fixed points are at (0,0), (3,0), (0,3) and (1,1).

a) (16 pts) For each fixed point, determine how many stable and unstable modes there are. Taking the gradients of the right hand gives

$$\begin{pmatrix} 3 - 2x_1 - 2x_2 & -2x_1 \\ -2x_2 & 3 - 2x_1 - 2x_2 \end{pmatrix}.$$

At the four fixed points this is $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} -3 & -6 \\ 0 & -3 \end{pmatrix}$, $\begin{pmatrix} -3 & 0 \\ -6 & -3 \end{pmatrix}$ and $\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$. The first matrix has two positive eigenvalues, so (0,0) has two unstable modes. The second matrix has two negative eigenvalues, so (3,0) has two stable modes. Ditto for (0,3). The last matrix has eigenvalues -3 and 1, so it has one stable and one unstable mode, and the fixed point (1,1) is unstable.

b) (4 pts) Describe the possible long-term behavior of the system.

There are two stable fixed points, and the system will either approach (3,0) or (0,3) in the long run. In other words, the competition is to the death. If we start out with $x_1 > x_2$, then we will approach (3,0), with the second species going extinct. If we start out with $x_1 < x_2$, then we will approach (0,3), with the first species going extinct.

This behavior depends on the parameters of the system. If instead we had studied the system of equations

$$\frac{dx_1}{dt} = x_1(3 - 2x_1 - x_2); \qquad \frac{dx_2}{dt} = x_2(3 - x_1 - 2x_2)$$

we would have found that (1,1) was a stable fixed point and that $(0, \frac{3}{2})$ and $(\frac{3}{2}, 0)$ were unstable. In this modified system, the competition is not so fierce, and the two species can coexist, always approaching an equilibrium where the two species have the same population. In the initial system, they can't.

5. a) Find an orthogonal basis for the column space of $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & -2 \\ 4 & 5 & 7 \end{pmatrix}$, where we are using the standard inner product for $\mathbb{R}^4$ (and subspaces of $\mathbb{R}^4$).

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}. \text{ Note that } \langle \mathbf{y}_1 | \mathbf{y}_1 \rangle = 30.$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \mathbf{y}_1 \tfrac{\langle \mathbf{y}_1 | \mathbf{x}_2 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle} = \mathbf{x}_2 - \tfrac{30}{30} \mathbf{y}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}. \text{ Note that } \langle \mathbf{y}_2 | \mathbf{y}_2 \rangle = 4.$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \mathbf{y}_1 \tfrac{\langle \mathbf{y}_1 | \mathbf{x}_3 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle} - \mathbf{y}_2 \tfrac{\langle \mathbf{y}_2 | \mathbf{x}_3 \rangle}{\langle \mathbf{y}_2 | \mathbf{y}_2 \rangle} = \mathbf{x}_2 - \tfrac{30}{30} \mathbf{y}_1 - \tfrac{8}{4} \mathbf{y}_2 = \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}.$$

b) Find an orthogonal basis for the column space of $\begin{pmatrix} 1 & 0 \\ i & 1-i \\ i & -1-3i \\ 2 & -5 \end{pmatrix}$, using the standard inner product for $\mathbb{C}^4$.

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{pmatrix} 1 \\ i \\ i \\ 2 \end{pmatrix}. \text{ Note that } \langle \mathbf{y}_1 | \mathbf{y}_1 \rangle = 1(1) + (-i)(i) + (-i)(i) + 2(2) = 7.$$

$\mathbf{y}_2 = \mathbf{x}_2 - \mathbf{y}_1 \frac{\langle \mathbf{y}_1 | \mathbf{x}_2 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle} = \mathbf{x}_2 - \frac{-14}{7} \mathbf{y}_1 = \begin{pmatrix} 2 \\ 1+i \\ -1-i \\ -1 \end{pmatrix}$, since $\langle \mathbf{y}_1 | \mathbf{y}_2 \rangle = 1(0) + (-i)(1-i) + (-i)(-1-3i) + 2(-5) = -14$. (If you forgot to take the complex conjugates of the elements of $|\mathbf{y}_1\rangle$ when computing $\langle \mathbf{y}_1|$, you probably got nonsense.)