M346 Final Exam, May 19, 2009

1. For each of these collections $\mathcal{B}$ of vectors in a vector space V, indicate (with explanation) whether $\mathcal{B}$ is linearly independent, spans V, both, or neither.

a) In R^3, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}$.

b) In R^4, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 10 \\ 5 \\ 4 \end{pmatrix} \right\}$.

c) In $R_2[t]$, $\mathcal{B} = \{1 + 2t + 3t^2, 3 + 6t + 10t^2, 1 + 3t + 4t^2\}$.

d) In $R_3[t]$, $\mathcal{B} = \{1 + 2t + 3t^2, 3 + 6t + 10t^2, 1 + 3t + 4t^2\}$.

2. In $R_2[t]$, consider the bases $\mathcal{E} = \{1, t, t^2\}$ and $\mathcal{B} = \{2, 2t + 5, t^2 + 5t + 7\}$, and the linear transformation $L : R_2[t] \to R_2[t]$, $L\mathbf{p}(t) = \mathbf{p}(t) + \mathbf{p}'(t)$.

a) Find the change-of-basis matrices $P_{\mathcal{B}\mathcal{E}}$ and $P_{\mathcal{E}\mathcal{B}}$.

b) If $\mathbf{p}(t) = t^2 + 9t + 23$, find $[\mathbf{p}]_{\mathcal{B}}$.

c) Find $[L]_{\mathcal{E}}$ and $[L]_{\mathcal{B}}$.

3. (a) Find a 2×2 matrix A whose eigenvalues are -30 and 40 and whose corresponding eigenvectors are $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. [Hint: the final answer should only involve integers, although you may see some fractions along the way.]

(b) What are the eigenvalues of $A^2 - 10A$?

c) Compute $A^2 - 10A$. (No, you do NOT need a calculator to do this.)

4. a) Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 9 & -7 \\ 4 & -2 \end{pmatrix}$.

b) Compute $e^{i\pi A}$. [The final answer involves rational numbers with small denominators.]

5. Consider the system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 \\ \frac{dx_2}{dt} &= 2x_1 + x_2 + 2x_3 \\ \frac{dx_3}{dt} &= 3x_1 + 2x_2 + x_3 \end{aligned}$$

a) Find the general solution.

b) If $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, what is the limiting value of $\frac{x_1(t)}{x_2(t)}$ as $t \to \infty$?

6. Use the Gram Schmidt process to convert the following basis for a 3-dimensional subspace of R^4 into an orthonormal basis for that subspace.

$\mathbf{x}_1 = (1, 1, -1, 0)^T$, $\mathbf{x}_2 = (4, 5, 0, 4)^T$, $\mathbf{x}_3 = (-2, 3, -2, -7)^T$.

7. Find all least-squares solutions to the system of equations

$$\begin{aligned} x_1 + 2x_2 &= -3 \\ 3x_1 + 2x_2 &= 7 \\ 2x_1 - 2x_2 &= 14 \\ 4x_1 - x_2 &= 5 \end{aligned}$$

8. Consider a wave $f(x,t)$ on the interval $[0,1]$, with Dirichlet boundary conditions ($f(0,t) = f(1,t) = 0$ for all time), moving with velocity 1. The initial condition is $f(x,0) = \sin(\pi x) + \sin(2\pi x)$ and $\frac{\partial f}{\partial t}(x,0) = 0$.

a) Find $f(x,t)$ for all x and all t.

b) Sketch $f(x,t)$ at times $t = 1/4$, $t = 1/2$, $t = 1$, $t = 3/2$, and $t = 2$. I've sketched $f(0,t)$ on the board to get you started.