M427J: Differential Equations with Linear Algebra

$\boxed{\text{Review I}}$

* $\boxed{\text{First Order ODEs}}$

(1) Linear Equations

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases} \qquad (E1)$$

(A) If $p(t)$ and $g(t)$ are continuous on an open interval I and $t_0 \in I$, then there exists a unique solution to $(E1)$ in I.

The uniqueness and existance of $(E1)$ does $\boxed{\text{not}}$ depe on the initial value y_0.

(B) Method of integrating factor

Introduce $\sigma(t)$ that solves $\sigma'(t) = p(t)\sigma(t)$ so that we can rewrite the eqn in $(E1)$ as

$$(\sigma y)' = \sigma g$$

$$\Rightarrow \quad \sigma y = \int \sigma g \, dt + C$$

$$y = \frac{1}{\sigma}\left[\int \sigma g \, dt + C\right]$$

(2) Nonlinear Equations

$$\begin{cases} \frac{dy}{dt} = f(y, t) \\ y(t_0) = y_0 \end{cases} \qquad (E2)$$

(A) Let f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle: $\alpha < t < \beta$, $\gamma < y < \delta$, containing (t_0, y_0). Then there is an interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ in which a unique solution to (E2) exists.

The existance and uniqueness of (E2) depend on the initial value y_0.

(3) If the equation is separable, then it can be solve by integration.

$$M(x) + N(y)\frac{dy}{dx} = 0$$

$$\Rightarrow M(x)dx + N(y)dy = 0$$

$$\Rightarrow \int M(x)dx + \int N(y)dy = C$$

(C) If the equation is more complicated but exact

$$\begin{cases} M(x, y) + N(x, y)\frac{dy}{dx} = 0 \\ M_y = N_x \end{cases}$$

Then there exists $\psi(x,y)$ so that $\psi_x = M$, $\psi_y = N$

$$\psi_x + \psi_y \frac{dy}{dx} = \frac{d}{dx}\psi(x, y(x)) = 0$$

$$\Rightarrow \quad \psi(x, y(x)) = C$$

(I) Integrating factor can also be introduced here to make $\sigma M + \sigma N \frac{dy}{dx} = 0$ an exact eqn.

In this case

$$(\sigma M)_y = (\sigma N)_x$$

$$\Rightarrow \quad \sigma_y M - \sigma_x N = (N_x - M_y)\sigma \qquad (E3)$$

(a) If $\sigma = \sigma(x)$, (E3) $\Rightarrow$

$$\frac{d\sigma}{dx} = -\frac{N_x - M_y}{N}\sigma \qquad (E4)$$

If $\frac{N_x - M_y}{N}$ is only a function of x, (E4) can be solved to get $\sigma(x)$.

(b) If $\sigma = \sigma(y)$, (E3) $\Rightarrow$

$$\frac{d\sigma}{dy} = \frac{N_x - M_y}{M}\sigma \qquad (E5)$$

If $\frac{N_x - M_y}{M}$ is only a function of y, (E5) can be solved to find $\sigma(y)$

* $\boxed{\text{Second Order Linear Equations}}$

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases} \qquad (E6)$$

(A) If $p(t)$, $g(t)$, $q(t)$ are all continuous in an open interval I and $t_0 \in I$, then there is a unique solution to (E6) throughout I.

(B). If the equation is homogeneous that is $g(t) = 0$ then $y'' + p(t)y' + q(t)y = 0$ (E7)

superposition principle tells us that if y_1 and y_2 are two solutions of (E7), then

$$y(t) = C_1 y_1 + C_2 y_2 \qquad (E8)$$

is also a solution. Also if y_1 and y_2 are linearly independent ($W[y_1, y_2](t) \neq 0$), then (E8) includes every solution of (E7).

(C). If the equation has constant coefficients.

$$ay'' + by' + cy = 0 \qquad (E9)$$

$(a \neq 0)$

then we have simple methods to solve it.

we look for exponential solutions

$$y(t) = e^{rt}$$

so that (E9) implies

$$ar^2 + br + c = 0 \qquad (E10)$$

ⓐ if $b^2 - 4ac > 0$, $r_1 \neq r_2$, real

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

ⓑ if $b^2 - 4ac < 0$. $r_{1,2} = \lambda \pm i\mu$

$$y(t) = e^{\lambda t}(C_1 \cos \mu t + C_2 \sin \mu t)$$

ⓒ. if $b^2 - 4ac = 0$, $r_1 = r_2$ real.

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$