Review I

* First Order ODEs

(1) Linear Equations
\[
\begin{align*}
  y' + p(t)y &= g(t) \\
  y(t_0) &= y_0
\end{align*}
\]  
(E1)

(A) If \( p(t) \) and \( g(t) \) are continuous on an open interval \( I \) and \( t_0 \in I \), then there exists a unique solution to (E1) in \( I \).

The uniqueness and existence of (E1) does not depend on the initial value \( y_0 \).

(B) Method of integrating factor

Introduce \( \sigma(t) \) that solves \( \sigma'(t) = p(t)\sigma(t) \) so that we can rewrite the eqn in (E1) as

\[
(\sigma y)' = \sigma g
\]

\[\Rightarrow \sigma y = \int g \, dt + C\]

\[y = \frac{1}{\sigma} \left[ \int g \, dt + C \right]\]
(2) Nonlinear Equations
\[
\begin{aligned}
\frac{dy}{dt} &= f(y, t) \\
y(t_0) &= y_0.
\end{aligned}
\] (E2)

(A) Let \( f \) and \( \frac{df}{dy} \) be continuous in some rectangle \( a < t < b, \ r < y < s \), containing \( (t_0, y_0) \). Then there is an interval \( t_0 - h < t < t_0 + h \) contained in \( a < t < b \) in which a unique solution to \( (E2) \) exists.

The existence and uniqueness of \( (E2) \) depend on the initial value \( y_0 \).

(3) If the equation is separable, then it can be solved by integration.
\[
M(x) + N(y) \frac{dy}{dx} = 0
\]
\[
\Rightarrow M(x) dx + N(y) dy = 0
\]
\[
\Rightarrow \int M(x) dx + \int N(y) dy = C
\]

(c) If the equation is more complicated but exact
\[
M(x, y) + N(x, y) \frac{dy}{dx} = 0
\]
\[
\begin{cases}
My = Nx
\end{cases}
\]
Then there exists \( f(x, y) \) so that \( f_x = M, f_y = N \)

\[
4 x + 4 y \frac{dy}{dx} = \frac{d}{dx} f(x, y(x)) = 0
\]

\[
\Rightarrow f(x, y(x)) = C
\]

\( \sigma \) Integrating factor can also be introduced here to make \( \sigma M + \sigma N \frac{dy}{dx} = 0 \) an exact eqn.

In this case

\[
(\sigma M)_y = (\sigma N)_x
\]

\[
\Rightarrow \sigma y M - \sigma x N = (N_x - M_y) \sigma \quad (E3)
\]

(a) If \( \sigma = \sigma(x) \), \( (E3) \Rightarrow \)

\[
\frac{d\sigma}{dx} = - \frac{N_x - M_y}{N} \sigma \quad (E4)
\]

If \( \frac{N_x - M_y}{N} \) is only a function of \( x \), \( (E4) \) can be solved to get \( \sigma(x) \).

(b) If \( \sigma = \sigma(y) \), \( (E3) \Rightarrow \)

\[
\frac{d\sigma}{dy} = \frac{N_x - M_y}{M} \sigma \quad (E5)
\]

If \( \frac{N_x - M_y}{M} \) is only a function of \( y \), \( (E5) \) can be solved to find \( \sigma(y) \).
Second Order Linear Equations

\[
\begin{align*}
\begin{cases}
y'' + p(t)y' + q(t)y = g(t) \\
y(t_0) = y_0, \quad y'(t_0) = y'_0
\end{cases}
\end{align*}
\tag{E6}
\]

(A) If \( p(t), q(t), g(t) \) are all continuous in an open interval \( I \) and \( t_0 \in I \), then there is a unique solution to (E6) throughout \( I \).

(B) If the equation is homogeneous that is \( g(t) = 0 \) then

\[
y'' + p(t)y' + q(t)y = 0 \quad \tag{E7}
\]

superposition principle tells us that if \( y_1 \) and \( y_2 \) are two solutions of (E7), then

\[
y(t) = C_1y_1 + C_2y_2 \quad \tag{E8}
\]

is also a solution. Also if \( y_1 \) and \( y_2 \) are linearly independent (\( W[y_1, y_2; i(t)] \neq 0 \)), then (E8) includes every solution of (E7).
(c). If the equation has constant coefficients.

\[ ay'' + by' + cy = 0 \quad (E9) \]
\[ (a \neq 0) \]

then we have simple methods to solve it.
we look for exponential solutions

\[ y(t) = e^{rt} \]
so that (E9) implies

\[ ar^2 + br + c = 0 \quad (E10) \]

(a) if \( b^2 - 4ac > 0 \), \( r_1 \neq r_2 \), real

\[ y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \]

(b) if \( b^2 - 4ac < 0 \), \( r_{1,2} = \lambda \pm i\mu \)

\[ y(t) = e^{\lambda t} (C_1 \cos \mu t + C_2 \sin \mu t) \]

(c) if \( b^2 - 4ac = 0 \), \( r_1 = r_2 \) real.

\[ y(t) = C_1 e^{rt} + C_2 e^{rt} \]