M427J: Differential Equations with Linear Algebra

Review 2

1. Nonhomogeneous 2nd-order Equations

Consider $y'' + p(t)y' + q(t)y = g(t)$

solution $Y(t) = \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{\text{solution of the corresponding homogeneous eqn}} + Y_p(t)$

$\uparrow$ one particular solution of nonhomogeneous eqn

① Method of Undetermined Coefficients

A). main principle: guess Y_p the same format of $g(t)$

B). for $g(t) = \cos\beta t$ or $\sin\beta t$, guess $A\cos\beta t + B\sin\beta t$.

C). for $g(t) = P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$, guess full polynomial $Y_p = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$

D) If the guess is a homogeneous solution, multiply the guess by t.

E) plug the particular solution Y_p into the original eqn and determine the coefficients.

② $\boxed{\text{Variation of Parameters}}$

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

Let y_1 and y_2 be two independent solutions of the corresponding homogeneous equation.

We look for the solution of the form

$$y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

$$\Rightarrow \begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(t) \end{cases}$$

$$\Rightarrow u_1'(t) = -\frac{y_2(t)\,g(t)}{W[y_1, y_2](t)}, \quad u_2'(t) = \frac{y_1(t)\,g(t)}{W[y_1, y_2](t)}$$

$$\Rightarrow u_1(t) = -\int \frac{y_2 g}{W}\,dt + C_1 \;; \quad u_2(t) = \int \frac{y_1 g}{W}\,dt + C_2 .$$

$$\Rightarrow Y_p(t) = -y_1 \int \frac{y_2 g}{W[y_1, y_2](t)}\,dt + y_2 \int \frac{y_1 g}{W[y_1, y_2](t)}\,dt$$

2. Series Solution of Second Order Linear Eqn

① Series Solution Near an Ordinary Points

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

A point x_0 for which $P(x_0) \neq 0$ is called an ordinary point of the ODE.

* Set $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$
* plug $y(x)$, $y'(x)$, $y''(x)$ into eqn
* get recurrence relation for coefficient.
* get general form of the coefficients from the recurrence relation. (If there is no general form of the coefficient compute 4 terms)
* compute the radius of convergence by using ratio test. (If no general form use Theorem)

② Series solution near a regular singular Point

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

If $P(x_0) = 0$

and $\lim_{x \to x_0} (x-x_0)\frac{Q(x)}{P(x)}$ is finite ← $p(x)$

$\lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)}$ is finite ← $q(x)$

Then x_0 is a regular singular point of the ODE.

① . Let $y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n$ $\boxed{a_0 \neq 0}$

②. plug $y(x)$, $y'(x)$ and $y''(x)$ into the eqn.

③ shift the index, organize the terms

④ $n=0 \Rightarrow$ get indicial eqn for r

$\Rightarrow$ solve indicial eqn for r

⑤ get recurrence relation for coefficients for each r

⑥. get general form of coefficients from the recurrence relation (compute 4 terms if there is no general form for the coefficients)

⑦ compute the radius of convergence by using ratiotest
(or use Theorem)

③ Euler eqn : $x^2 y'' + \alpha x y' + \beta y = 0$ (α, β constant)
try $y = x^r$

$\Rightarrow \quad r^2 + (\alpha - 1) r + \beta = 0$

$$r_{1,2} = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}.$$

$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$ (for $r_1 \neq r_2$ real)