M427J: Differential Equations with Linear Algebra

Review 2

1. Nonhomogeneous 2nd-order Equations

Consider $y'' + p(t)y' + q(t)y = g(t)$

solution $y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

$\uparrow$

Solution of the one particular corresponding homogeneous eqn

nonhomogeneous eqn

1. Method of Undetermined Coefficients

A). main principle: guess $y_p$ the same format of $g(t)$

B). for $g(t) = \cos bt$ or $\sin bt$, guess $A\cos bt + B\sin bt$

C). for $g(t) = P(x) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$, guess full polynomial $y_p = A_0 t^n + A_1 t^{n-1} + \cdots + A_{n-1} t + A_0$

D). If the guess is a homogeneous solution, multiply the guess by $t$.  

$\ominus$
F) plug the particular solution $y_p$ into the original eqn and determine the coefficients.

**Variation of Parameters**

$$y'' + p(t)y' + q(t)y = g(t)$$

Let $y_1$ and $y_2$ be two independent solutions of the corresponding homogeneous equation.

We look for the solution of the form

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$\Rightarrow \begin{cases} 
  u_1'y_1 + u_2'y_2 = 0 \\
  u_1'y_1' + u_2'y_2' = g(t)
\end{cases}$$

$$\Rightarrow \begin{cases} 
  u_1(t) = -\frac{y_2(t)g(t)}{W[y_1,y_2](t)} \\
  u_2(t) = \frac{y_1(t)g(t)}{W[y_1,y_2](t)}
\end{cases}$$

$$\Rightarrow \begin{align*}
  y_1(t) &= -\int \frac{y_2g}{W} \, dt + C_1 \\
  y_2(t) &= \int \frac{y_1g}{W} \, dt + C_2
\end{align*}$$

$$\Rightarrow y_p(t) = -y_1\int \frac{y_2g}{W[y_1,y_2](t)} \, dt + y_2\int \frac{y_1g}{W[y_1,y_2](t)} \, dt$$
2. **Series Solution of Second Order Linear Eqn**

1. **Series Solution Near an Ordinary Point**

   \[ P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0 \]

   A point \( x_0 \) for which \( P(x_0) \neq 0 \) is called an ordinary point of the ODE.

   * Set \( y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \)

   * Plug \( y(x), y'(x), y''(x) \) into eqn

   * Get recurrence relation for coefficient

   * Get general form of the coefficients from the recurrence relation. (If there is no general form of the coefficients, compute 4 terms)

   * Compute the radius of convergence by using ratio test. (If no general form, use theorem)

2. **Series Solution Near a Regular Singular Point**

   \[ P(x)y'' + Q(x)y' + R(x)y = 0 \]
If \( P(x_0) = 0 \)

and \( \lim_{x \to x_0} \left( x-x_0 \right) \frac{Q(x)}{P(x)} \) is finite

Then \( x_0 \) is a regular singular point of the ODE.

1. Let \( y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n \) \( \left( a_0 \neq 0 \right) \)

2. plug \( y(x) \), \( y'(x) \) and \( y''(x) \) into the eqn.

3. shift the index, organize the terms

4. \( n=0 \Rightarrow \) get indicial eqn for \( r \)

   \( \Rightarrow \) solve indicial eqn for \( r \)

5. get recurrence relation for coefficients for each \( r \)

6. get general form of coefficients from the recurrence relation (compute 4 terms if there is no general form for the coefficients)
1. Compute the radius of convergence by using ratio test (or use Theorem)

2. Euler eqn: \( x^2 y'' + 2xy' + \beta y = 0 \) \((\beta \text{ constant})\)
   
   Try \( y = x^r \)
   
   \[ r^2 + (2-1)r + \beta = 0 \]
   
   \[ r_{1,2} = \frac{-(2-1) \pm \sqrt{(2-1)^2 - 4\beta}}{2} \]

   \[ y(x) = C_1 x^{r_1} + C_2 x^{r_2} \quad \text{(for } r_1 \neq r_2 \text{ real)} \]