Chapter 3. **Linear System of Equation**

1. **Eigenvalue Problems**

   \[ AX = \lambda X \quad (X \neq 0) \quad (E1) \]
   \[ (A - \lambda I)X = 0 \quad (E2) \]
   1. For nonzero \( X \), \( \text{det}(A - \lambda I) = 0 \)
      \[ \Rightarrow \text{solve for } \lambda \]
   2. Plug \( \lambda \) into \( (E2) \), get corresponding \( X \).

2. **General** \( X' = AX \)

   Suppose \( A \) is a 5x5 matrix whose eigenvalues are \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) and \( \lambda_5 \), where \( \lambda_2 = \bar{\lambda}_3 \), \( \lambda_4 = \lambda_5 \).
   Find the solution for \( X' = AX \).

   \[
   \lambda_1 \rightarrow \tilde{z}^{(1)} \rightarrow X^{(1)} = \tilde{z}^{(1)} e^{\lambda_1 t}
   \]
\[ r_2 = r_3 \quad \rightarrow \quad z^{(2)} \quad \rightarrow \quad x^{(2)} = z^{(2)}e^{rt} = \text{complex} = u(t) + iv(t) \]

\[ r_4 = r_5 \quad \rightarrow \quad z^{(4)} \quad \rightarrow \quad x^{(4)} = \tilde{z}^{(4)}e^{rt} \]

\[ x^{(5)} = \tilde{z}^{(4)}te^{rt} + \eta e^{rt} \]

where \( \eta \) is determined by

\[(A - r+I)\eta = z^{(4)} \]

So

\[ x(t) = C_1 \tilde{z}^{(4)}e^{rt} + C_2 u(t) + C_3 v(t) \]

\[ + C_4 \tilde{z}^{(4)}e^{rt} + C_5 [\tilde{z}^{(4)}te^{rt} + \eta e^{rt}] \]

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Chapter 5. \underline{Partial Differential Equations}

* Two-Point-Boundary Value Problem

BVP: no uniqueness sometime

* Eigenvalue Problems

\[ y''(x) + \lambda y(x) = 0 \]

\[ y(0) = 0, \quad y(L) = 0 \]

\[ \Rightarrow \lambda_n = \left( \frac{n\pi}{L} \right)^2, \quad y_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, \ldots \]
\[ y''(x) + \lambda y(x) = 0 \]

\( \begin{cases} y'(0) = 0, \ y'(a) = 0 \end{cases} \)

\[ \Rightarrow \lambda_n = \left( \frac{n\pi}{L} \right)^2, \quad y_n(x) = \cos \frac{n\pi x}{L}, \quad n = 0, 1, 2 \ldots \]

**Method**: How to discuss \( \lambda \)

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*Eigenfunction Set*

\[ \{ 1, \{ \cos \frac{n\pi x}{L} \}_{n=1}^{\infty}, \{ \sin \frac{n\pi x}{L} \}_{n=1}^{\infty} \} \]

Any two distinct eigenfunctions in the set are orthogonal. \( \Rightarrow \) inner product is zero.

If not distinct:

\[ \int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = L \quad n = m \]

\[ \int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = L \quad n = m \]

\[ \int_{-L}^{L} 1 \cdot 1 \, dx = 2L \]

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The series $\sum_n a_n \cos \frac{nx}{L} + \sum_n b_n \sin \frac{nx}{L}$ converge to $f(x)$ at all points where $f(x)$ is continuous and $f(x) \pm f'(x)$ at all points where $f(x)$ is discontinuous.

Convergence Theorem

$A_n = \frac{1}{L} \int_0^L f(x) \cos \frac{nx}{L} \, dx$

$B_n = \frac{1}{L} \int_0^L f(x) \sin \frac{nx}{L} \, dx$

where

$A_0 = \frac{1}{2L} \int_0^L f(x) \, dx$

$B_n = \frac{1}{2} \left[ f(x) \cos \frac{nx}{L} \right]_0^L + b_n \sin \frac{nx}{L}$

$a_n = A_n + i B_n$

$b_n = A_n - i B_n$

Fourier Series

\[
\int_0^L f(x) \cos \frac{nx}{L} \, dx = \frac{A_n}{\cos \frac{nx}{L}}
\]

\[
\int_0^L f(x) \sin \frac{nx}{L} \, dx = \frac{B_n}{\sin \frac{nx}{L}}
\]
Seperation of Variables

1. Assume \( u(x,y) = X(x)Y(y) \) (or \( u(x,t) = X(x)T(t) \))
   \( u_x = X'Y, \ u_{xx} = X''Y, \ u_y = XY', \ u_{yy} = XY'' \)

2. Plug into original eqn to get 2 separate eqns for \( x \) or \( y \) only, respectively.
   
   eqn for \( x \) only \hspace{1cm} \{ \text{both of them contain } \lambda \}. Which one is the eigenvalue problem?
   
   eqn for \( y \) only

3. Separate BCs
   Separate homogeneous BCs only
   
   leave the nonhomogeneous condition alone.

4. Solve eigenvalue problem first
   \( \text{eqn + 2 homogeneous BCs} \)
   
   \( \Rightarrow \) get eigenvalues and eigenfunctions

5. plug eigenvalues into the 2nd eqn and solve it.
6. the general solution is

\[ u(x, y) = \sum_{n=0}^{\infty} C_n U_n = \sum_{n=0}^{\infty} C_n X_n(x) Y_n(y) \]

7. Use IC or the nonhomogeneous BC to determine the arbitrary constants \( C_n \).