M42TJ: Differential Equations with Linear Algebra

$\boxed{\text{Review 3}}$

Chapter 3. $\boxed{\text{Linear System of Equation}}$

① $\boxed{\text{Eigenvalue Problems}}$

$$AX = \lambda X \quad (X \neq 0) \qquad \text{(E1)}$$

$$(A-\lambda I)X = 0 \qquad \text{(E2)}$$

① for nonzero X, $\det(A-\lambda I) = 0$

$\Rightarrow$ solve for λ

② plug λ into (E2), get corresponding X.

② $\boxed{\text{General } X' = AX}$

Suppose A is a 5×5 matrix whose eigenvalues are r_1, r_2, r_3, r_4 and r_5, where $r_2 = \overline{r_3}$, $r_4 = r_5$. find the solution for $X' = AX$.

$$r_1 \longrightarrow \xi^{(1)} \longrightarrow X^{(1)} = \xi^{(1)} e^{r_1 t}$$

$r_2 = \overline{r_3} \longrightarrow \xi^{(2)} \longrightarrow X^{(2)} = \xi^{(2)} e^{r_2 t}$ ← complex

$= u(t) + i v(t)$

$r_4 = r_5 \longrightarrow \xi^{(4)} \longrightarrow X^{(4)} = \xi^{(4)} e^{r_4 t}$

$X^{(5)} = \xi^{(4)} t e^{r_4 t} + \eta e^{r_4 t}$

where η is determined by

$(A - r_4 I)\eta = \xi^{(4)}$

So $X(t) = c_1 \xi^{(1)} e^{r_1 t} + c_2 u(t) + c_3 v(t)$
$+ c_4 \xi^{(4)} e^{r_4 t} + c_5 [\xi^{(4)} t e^{r_4 t} + \eta e^{r_4 t}]$

Chapter 5. Partial Differential Equations

* **TWO-Point-Boundary Value Problem**

  BVP: no uniqueness sometime

* **Eigenvalue Problems**

$$\begin{cases} y''(x) + \lambda y(x) = 0 \\ y(0) = 0, \quad y(L) = 0 \end{cases}$$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \cdots$

$$\begin{cases} y''(x) + \lambda y(x) = 0 \\ y'(0) = 0 \ , \ y'(L) = 0 \end{cases}$$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$, $y_n(x) = \cos\frac{n\pi x}{L}$, $n = 0, 1, 2 \ldots$

Method: How to discuss λ

* **Eigenfunction Set**

$$\left\{ 1 \ , \ \left\{\cos\frac{n\pi x}{L}\right\}_{n=1}^{\infty} \ , \ \left\{\sin\frac{n\pi x}{L}\right\}_{n=1}^{\infty} \right\}$$

any two distinct eigenfunctions in the set are orthogonal. $\Rightarrow$ inner product is zero.

if not distinct:

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} \, dx = L \qquad n = m$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} \, dx = L \qquad n = m$$

$$\int_{-L}^{L} 1 \cdot 1 \, dx = 2L$$

* **Fourier Series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad (*)$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n=1,2,\cdots$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n=1,2,\cdots$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

* **Convergence Theorem**

The series $(*)$ converge to $f(x)$ at all points where $f(x)$ is continuous and $\frac{f(x+)+f(x-)}{2}$ at all points where f is discontinuous.

* **Cosine and Sine Series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

↑ even

with $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, $n=0,1,2\cdots$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

↑ odd

with $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$, $n=1,2\cdots$

* $\boxed{\text{Seperation of Vatiable}}$

① assume $u(x,y) = X(x)\,Y(y)$ (or $u(x,t) = X(x)\,T(t)$)

$u_x = X'Y$, $u_{xx} = X''Y$, $u_y = XY'$, $u_{yy} = XY''$

② plug into original eqn to get 2 separate eqns for x or y only, respectively.

eqn for x only
eqn for y only

} both of them contain λ. Which one is the eigenvalue problem?

③ Separate BCs

Separate $\boxed{\text{homogeneous}}$ BCs only

leave the nonhomogeneous condition alone.

④ Solve eigenvable problem first

$\boxed{\text{eqn + 2 homogeneous BCs}}$

$\Rightarrow$ get eigenvalues and eigenfunctions

⑤ plug eigenvalues into the 2nd eqn and solve it.

⑥ the general solution is

$$u(x,y) = \sum_{n=0}^{\infty} C_n u_n = \sum_{n=0}^{\infty} C_n X_n(x)\, Y_n(y)$$

$\boxed{\text{or } 1}$

⑦ use IC or the nonhomogeneous BC to determine the arbitrary constants C's.