HW 3, Section 1.10, Problem 8

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Solve the IVP and determine how the interval in which the solution exists depends on the initial value y_0

$$y' = 2ty^2, \qquad y(0) = y_0.$$

We easily get an implicit solution since the equation is separable:

$$\int \frac{dy}{y^2} = \int 2tdt \implies \frac{-1}{y} = t^2 + C.$$

Next we need to answer the part of the question concerned with where the solution actually exists. We need to recall the theorem from the book, presented in lecture. The theorem is for a general first order, nonlinear ODE of the form

$$y' = f(t, y), \qquad y(t_0) = y_0.$$

The theorem says: *Let f and $\partial_y f$ be continuous in the rectangle $R : t_0 \leq t \leq t_0 + a,\ y_0 - b \leq y \leq y_0 + b$. If we define*

$$M = \max_{(t,y) inR} |f(t, y)|$$

and set $\alpha = \min\{a, b/M\}$ then the IVP has a unique solution $y(t)$ on the interval $t_0 \leq t \leq t_0 + \alpha$.

We want to use this theorem to say something about the IVP from problem 8. Note that in this case,

$$f(t, y) = 2ty^2, \qquad \partial_y f(t, y) = 4ty$$

both of which are continuous in the entire ty plane (polynomials are always continuous). In this particular case, then, the "rectangle" in the ty plane referred to in the theorem is actually the half plane $\{(t, y)|t \geq t_0 = 0\}$ (if you like, the first and fourth quadrant of the ty plane including the line $t = 0$). However, the time interval of existence from the theorem is $0 \leq t \leq \alpha$. But, α depends on a, b, and M, and in particular, the value M depends on the maximum of $|f(t, y)|$ in the rectangle defined by a and b. We can't use the theorem when we're talking about the entire half plane.

So, the best way to continue with the problem is to proceed with **fixed but arbitrary** a and b. Then we will put our answer, α, in terms of M (which we can find when a and b are given) and a and b. Note that if you were tempted to take $a = \infty$, since f and $\partial_y f$ are continuous everywhere, you'd find that M would also be infinite, which is not good.

To finish the problem, we need to find the maximum of the function $f(t, y) = 2ty^2$ in the rectangle $[0, a] \times [y_0 - b, y_0 + b]$, which is simply

$$M = 2a(y_0 + b)^2$$

if $y_0 > 0$, or

$$M = 2a(y_0 - b)^2$$

if $y_0 < 0$. Then we can use the theorem to get a unique solution to the ODE in the interval $0 \leq t \leq \alpha$, where

$$\alpha = \min\left\{a, \frac{b}{2a(y_0 \pm b)^2}\right\}.$$