One-to-One Functions & Onto Functions

Official In-the-book Definitions: Let $F$ be a function from a set $X$ to a set $Y$.

$F$ is one-to-one (or injective) $\iff$ For every $u$ and $v$ in $X$,

\[\text{If } F(u) = F(v), \text{ Then } u = v\]

Also,

$F$ is one-to-one (or injective) $\iff$ For every $u$ and $v$ in $X$,

\[\text{If } u \neq v, \text{ Then } F(u) \neq F(v) .\]

$F$ is onto (or surjective) $\iff$ For every element $y \in Y$,

there exists some $x \in X$ such that $F(x) = y$.

$F$ is a one-to-one correspondence (or a bijection) from $X$ to $Y$

$\iff$ $F : X \rightarrow Y$ is both a one-to-one function and an onto function.

Memorize the above definitions for their use in writing proofs, but a more intuitive definition of these terms is useful and is as follows:

Let $f : X \rightarrow Y$ be a function.

Function $f$ is … \[
\begin{cases}
\text{onto} & \text{if each element of } Y \text{ is the at least one element of } X \\
\text{one-to-one} & \text{if each element of } Y \text{ is the at most one element of } X \\
\text{one-to-one and onto} & \text{exactly one element of } X
\end{cases}
\]

If $f : X \rightarrow Y$ is one-to-one and onto, then the inverse function $f^{-1} : Y \rightarrow X$ exists and $f^{-1}(y) = x$ if and only if $f(x) = y$, for all $x$ in $X$ and all $y$ in $Y$. 
Proof Design I for Proving Function $F$ is One-to-One:

Function $F: X \rightarrow Y$ is given.

To Prove: Function $F$ is a one-to-one function.

Proof: Suppose that $u$ and $v$ are any two elements of $X$ such that

$F(u) = F(v)$. [We need to show that $u = v$.]

\[\ldots\]

\[\ldots\] (Using the formula defining $F(x)$ or some other properties
\[\ldots\] of the function $F$ we derive simpler and simpler
\[\ldots\] equations eventually arriving at “$u = v$.”)

$\therefore u = v$.

[ $\therefore \forall u, v \in X$, If $F(u) = F(v)$, Then $u = v$.]

$\therefore F$ is one-to-one, by Direct Proof, by Direct Proof. Q E D

Proof Design II for Proving Function $F$ is One-to-One:

Function $F: X \rightarrow Y$ is given.

To Prove: Function $F$ is a one-to-one function.

Proof: Suppose that $u$ and $v$ are any two elements of $X$ such that

$u \neq v$. [We need to show that $F(u) \neq F(v)$.]

\[\ldots\]

\[\ldots\] (This is often accomplished using a proof-by-contradiction,
\[\ldots\] but sometimes it can be shown directly that $F(u) \neq F(v)$.)

\[\ldots\]

$\therefore F(u) \neq F(v)$.

[ $\therefore \forall u, v \in X$, If $F(u) = F(v)$, Then $u = v$, by contraposition.]

$\therefore F$ is one-to-one by Direct Proof. Q E D
Proof Design for Proving that Function $F$ is Onto:

Function $F: X \rightarrow Y$ is given.

To Prove: Function $F$ is an onto function.

Proof: Suppose $y$ is any element in $Y$.

[ We need to show that there is some $x$ in $X$ with $F(x) = y$. ]

(Note: In a workspace, and before the writing of the proof has begun, the equation $F(x) = y$ is manipulated in order to solve for $x$ in terms of $y$ deriving a formula: $x = \text{“Formula in terms of } y\text{”}$. Use this formula to define the correct pre-image $x$ for the selected $y$ at the start.)

Let $x = \text{“Formula in terms of } y\text{”}$

Then, $F(x) = (\text{the complicated expression obtained by replacing } x \text{ by the “Formula in terms of } y\text{”}) = \ldots \text{(simplifications)} \ldots = y$.

[ $\therefore \forall y \text{ in } Y, \text{ there exists an element } x \text{ in } X \text{ such that } F(x) = y$. ]

$\therefore F$ is onto, by Direct Proof. Q E D

Proof Design to Prove that $F$ is a One-to-One Correspondence (or Bijection):

Function $F: X \rightarrow Y$ is given.

To Prove: $F$ is a One-to-One Correspondence.

Proof:

Part I: [ Prove $F$ is one-to-one.] $\ldots \ldots \therefore F$ is one-to-one by Direct Proof.

Part II: [ Prove $F$ is onto.] $\ldots \ldots \therefore F$ is onto by Direct Proof.

$\therefore F$ is one-to-one and onto.

$\therefore F$ is a one-to-one correspondence. Q E D