PROOF THAT \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \) using the Well-Ordering Principle of \( \mathbb{Z} \).

To Prove: For all integers \( n \geq 1 \), \( 1+2+\ldots+n = \frac{n(n+1)}{2} \).

Proof: Suppose, by way of contradiction, that there exists an integer \( n_0 \) such that \( n_0 > 1 \) and
\[
1+2+\ldots+n_0 \neq \frac{n_0(n_0+1)}{2}.
\]

Let \( S = \{ \text{all integers } x \text{ such that } x \geq 1 \}
\text{ and } 1+2+\ldots+x \neq \frac{x(x+1)}{2} \} \).

By supposition, \( n_0 > 1 \) and \( 1+2+\ldots+n_0 \neq \frac{n_0(n_0+1)}{2} \).

\[ \therefore \text{ Integer } n_0 \text{ is in set } S, \text{ and so set } S \text{ is not empty.} \]
\[ \therefore \text{ Condition 1 of the Well-Ordering Principle is satisfied.} \]

By definition of set \( S \), for every \( x \in S, x \geq 1 \).
\[ \therefore \text{ Condition 2 of the Well-Ordering Principle is satisfied.} \]

\[ \therefore \text{ By the Well-Ordering Principle, set } S \text{ has a least element, } m. \]
\[ \therefore m > 1 \text{ and } 1+2+\ldots+m \neq \frac{m(m+1)}{2} \text{ and } m \text{ is the least positive integer with both of these properties.} \]
Note that this means that, if \( x \) is an integer such that \( x \geq 1 \) and \( x \) is NOT in set \( S \), then
\[
1 + 2 + \cdots + x = \frac{x(x+1)}{2}, \text{ or otherwise } x \text{ would be in set } S. \]

Now, \( m - 1 < m \).
\[
\therefore \ m - 1 \text{ is not in set } S \text{ because } m \text{ is the least element in set } S.
\]

[We need to show that \( m - 1 \geq 1 \), that is, that \( m \geq 2 \) which follows if we show that \( m \neq 1 \)]

Let \( n = 1 \). Then, \( 1 + 2 + \cdots + n = 1 \) and \( \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1 \).
\[
\therefore \ \text{For } n = 1, \ 1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \text{ Thus, } n = 1 \text{ is not in set } S, \text{ so } m \neq n = 1, \text{ because } 1 + 2 + \cdots + m \neq \frac{m(m+1)}{2}.
\]
\[
\therefore \ m > 2.
\]
\[
\therefore \ m - 1 \geq 1
\]

Since \( m - 1 \geq 1 \) and \( m - 1 \) is NOT in set \( S \),
\[
\therefore \ 1 + 2 + \cdots + (m - 1) = \frac{(m-1)(m)}{2}. \text{ [See Note Above]}
\]
\[
\therefore \ (1 + 2 + \cdots + (m-1)) + m = \frac{(m-1)(m)}{2} + m, \text{ by substitution}
\]
\[
\therefore \ 1 + 2 + \cdots + (m-1) + m = \frac{(m-1)m + 2m}{2} = \frac{(m-1+2)m}{2},
\]
\[
\therefore \ 1 + 2 + \cdots + m = \frac{m(m+1)}{2}, \text{ which contradicts the fact that } 1 + 2 + \cdots + m \neq \frac{m(m+1)}{2}.
\]
\[
\therefore \ \text{By proof-by-contradiction, for all integers } n \geq 1,
\]
\[
1 + 2 + \cdots + n = \frac{n(n+1)}{2}.
\]
\[\text{ QED} \]