The Derivation of the TRIG INTEGRAL REDUCTION FORMULAS

Here, we provide derivations of the following formulas:

(\(k\) and \(t\) represent positive integers where \(k\) is even and \(t\) is odd and \(t > 1\).)

I. \(\int \tan^k x \, dx = \frac{1}{k-1} \tan^{k-1} x - \int \tan^{k-2} x \, dx\)

and

II. \(\int \sec^t x \, dx = \frac{1}{t-1} \tan x \sec^{t-2} x + \frac{t-2}{t-1} \int \sec^{t-2} x \, dx\).

Derivation of I:
Recall that \(k\) is an even positive integer. Thus, \(k \geq 2\) and \(k-2 \geq 0\).

First note that \(\int \tan^{k-2} x \sec^2 x \, dx = \int u^{k-2} \, du\)

When \(u = \tan x\) and \(du = \sec^2 x \, dx\).

Thus, \(\int \tan^k x \sec^2 x \, dx = \int u^{k-2} \, du = \frac{1}{k-1} u^{k-1} + C = \frac{1}{k-1} \tan^{k-1} x + C\).

Using the fact that \(\tan^2 x = (\sec^2 x - 1)\), we have that

\[\int \tan^k x \, dx = \int (\tan^{k-2} x)(\tan^2 x) \, dx = \int (\tan^{k-2} x)(\sec^2 x - 1) \, dx\]

\[= \int (\tan^{k-2} x)(\sec^2 x) \, dx - \int \tan^{k-2} x \, dx\]

\[= \frac{1}{k-1} \tan^{k-1} x - \int \tan^{k-2} x \, dx\]

Therefore, \(\int \tan^k x \, dx = \frac{1}{k-1} \tan^{k-1} x - \int \tan^{k-2} x \, dx\) when \(k\) is an even positive integer.
Derivation of II:

\[ \int \sec^t x \, dx = \frac{1}{t-1} (\tan x)(\sec^{t-2} x) + \frac{t-2}{t-1} \int \sec^{t-2} x \, dx \]

Recall that \( t \) is an odd integer such that \( t > 1 \). Since \( t \geq 3 \), \( t-2 \geq 1 \).

First note that \( \int \sec^t x \, dx = \int (\sec^{t-2} x)(\sec^2 x) \, dx \).

We apply integration-by-parts with

\[ u = \sec^{t-2} x, \quad dv = (\sec^2 x) \, dx, \]

\[ du = (t-2)(\sec^{t-3} x)(\sec x)(\tan x) \, dx, \quad v = \tan x. \]

Thus, \( \int \sec^t x \, dx = (\sec^{t-2} x)(\tan x) - (t-2) \int (\tan^2 x)(\sec^{t-2} x) \, dx \)

\[ = (\tan x)(\sec^{t-2} x) - (t-2) \int (\sec^3 x - 1)(\sec^{t-2} x) \, dx \]

\[ = (\tan x)(\sec^{t-2} x) - (t-2) \left[ \int \sec^t x \, dx - \int (\sec^{t-2} x) \, dx \right]. \]

Thus, \( \int \sec^t x \, dx = (\tan x)(\sec^{t-2} x) - (t-2) \int \sec^t x \, dx + (t-2) \int \sec^{t-2} x \, dx \).

We solve for \( \int \sec^t x \, dx \) algebraically, first by adding \( (t-2) \int \sec^t x \, dx \) to both sides of this equation. Note that \( 1 + (t-2) = (t-1) \).

Thus, \( (t-1) \int \sec^t x \, dx = (\tan x)(\sec^{t-2} x) + (t-2) \int \sec^{t-2} x \, dx \).

Dividing both sides of this equation by \( t-1 \), we have

\[ \int \sec^t x \, dx = \frac{1}{t-1} (\tan x)(\sec^{t-2} x) + \frac{t-2}{t-1} \int \sec^{t-2} x \, dx, \]

and Reduction Formula II has been derived.