THE LIMIT OF A FUNCTION $z = f(x,y)$ OF 2 VARIABLES

let $z = f(x,y)$ be a function of two variables and let $(a,b)$ be a particular ordered pair in $\mathbb{R}^2$.

We discuss here what it means to write

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

and to say

"the limit of $f(x,y)$, as $(x,y)$ approaches $(a,b)$, exists and is the number $L$.

Loosely speaking, we are saying here that,

as the point $(x,y)$ in the $xy$ plane gets ever closer to $(a,b)$, the function value $f(x,y)$ at the point $(x,y)$ gets ever closer to $L$.

The official definition is this:

The limit $\lim_{(x,y) \to (a,b)} f(x,y)$ exists and is equal to $L$

if and only if,

for every number $\varepsilon > 0$, there is a $\delta > 0$ such that

whenever $(x,y)$ is a point in the domain $D$ of $f$ such that

$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, we will have $|f(x,y) - L| < \varepsilon$.

Using an arrow diagram, this definition is illustrated as follows:
Given any $\varepsilon > 0$, we can find a punctured $\delta$-disk about $(a, b)$, so small that...

when $(x, y)$ is here...

...that...

When $(x, y)$ is in the ball and $(x, y) \neq (a, b)$, the distance between $f(y, x)$ and $L$ is $< \varepsilon$.

This illustration can also be made by consulting the surface graph of the function $z = f(x, y)$, near the point $(2, 3)$.

**For example:** For $f(x, y) = 1 + x^2 + y^2$, $1 + x^2 + y^2$

$$\lim_{(x, y) \to (2, 3)} f(x, y) = \lim_{(x, y) \to (2, 3)} (1 + x^2 + y^2) = 14$$

**Definition:** If $\lim_{(x, y) \to (a, b)} f(x, y) = L$ and $L = f(a, b)$, then

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$ and we say $f$ is continuous at $(a, b)$. 

At $(2, 3)$...
AN EXAMPLE WHERE \( \lim_{(x,y) \to (y,0)} f(x,y) \) D.N.E.

Define \( f(x,y) = \begin{cases} 
40 & \text{if } y \geq 0 \\
-10 & \text{if } y < 0 
\end{cases} \)

\( f(25,10) = 40 \) \( \sin \alpha \geq 0 \)

\( f(5,-20) = -10 \)

\( \lim_{(x,y) \to (y,0)} f(x,y) \) D.N.E.

\( \lim_{(x,y) \to (5,20)} f(x,y) = 40 \)

\( \lim_{(x,y) \to (5,-20)} f(x,y) = -10 \)
A Limit Theorem: If \( \lim_{(x,y) \to (a,b)} f(x,y) = L \) and 
\( g(t) \) is a function of one variable such that \( g \) is continuous at \( t = L \), then 
\( g \) of \((x,y) = g(f(x,y))\) is a function of 2 variables 
and 
\( \lim_{(x,y) \to (a,b)} g(f(x,y)) = g(L) \).

For example, since \( \lim_{(x,y) \to (2,3)} (1 + x^2 + y^2) = 14 \),
\( \lim_{(x,y) \to (2,3)} \sqrt{1 + x^2 + y^2} = \sqrt{14} \),
\( \lim_{(x,y) \to (2,3)} e^{1 + x^2 + y^2} = e^{14} \),
\( \lim_{(x,y) \to (2,3)} \ln(1 + x^2 + y^2) = \ln 14 \),
and \( \lim_{(x,y) \to (2,3)} \cos(1 + x^2 + y^2) = \cos 14 \).

FACT: When \( f(x,y) \) is a polynomial function [like \( z = 1 + x^2 + y^2 \)]
or when \( f(x,y) \) is a rational function [like \( f(x,y) = \frac{x^2 + xy + y^2}{x^2 + y^2} \)],
f is continuous at every point \((x,y)\) in its domain.
The following figure illustrates that \( \lim_{(x,y) \to (0,0)} 1 + x^2 + y^2 \) exists and \( \lim_{(x,y) \to (0,0)} 1 + x^2 + y^2 = 1. \)

Here, \( z = f(x,y) = 1 + x^2 + y^2 \)

The function \( z = f(x,y) = \frac{xy}{x^2 + y^2} \) is a function for which \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} \) does not exist.

Here is the graph of \( y = f(x,y) = \frac{xy}{x^2 + y^2} \)

The graph is rotated 90 degrees clockwise in the figure to the right.

When \( y = 0 \), \( f(x,0) = f(0,0) = \frac{0}{x^2} = 0 \)

When \( y = x \), \( f(x,x) = f(1,1) = \frac{x^2}{2x^2} = \frac{1}{2} \)

When \( y = -x \), \( f(x,-x) = f(1,-1) = \frac{-x^2}{2x^2} = -\frac{1}{2} \)

When \( x = 0 \), \( f(x,y) = f(0,y) = 0/\sqrt{y} = 0 \)

The limit \( L \) cannot equal \( \frac{1}{2} \) because every \( \delta \)-ball about \((0,0)\) contains points \((x,y)\) with \( z = \frac{1}{2} \), with \( z = -\frac{1}{2} \), all with \( z = 0 \).
The graph of

\[ f(x, y) = \frac{xy}{x^2 + y^2} \]

See Figure 6 on page 906 of Stewart's "Calculus, 8e" for a look at this graph from a different angle.
We can similarly conclude that $L$ cannot equal 0 or $-\frac{1}{2}$, or any other number.

Thus, \[ \lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2} \quad \text{D.N.E.} \quad \text{(Does not exist)} \]

Actually, there is an easier way to prove that \[ \lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2} \quad \text{does not exist.} \]

**FACT:** If \( \lim_{(x,y) \to (a,b)} f(x,y) \) exists and equals \( L \),

then \( \lim_{(x,y) \to (a,b)} f(x,y) = L \) and it must be the

same limit \( L \) for every path along any path to \((a,b)\) from any direction!

If there are two distinct paths \( C_1 \) and \( C_2 \) such that \( \lim_{(x,y) \to (a,b)} f(x,y) = L_1 \) and \( \lim_{(x,y) \to (a,b)} f(x,y) = L_2 \) and \( L_1 \neq L_2 \),

then \( \lim_{(x,y) \to (a,b)} f(x,y) \) does not exist.

This is similar to the Theorem in functions of one variable that says, if \( \lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x) \), then \( \lim_{x \to a} f(x) \) D.N.E.
Consider again the function 
\[ f(x, y) = \frac{xy}{x^2 + y^2} \]

We wish to show that \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} \) D.N.E.

In the \( xy \)-plane, we look at two paths by which \((x,y)\) can approach \((0,0)\), one along the line \( y = x \) and the other along the line \( y = -x \).

\[ \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} = \lim_{(x,x) \to (0,0)} \frac{x^2}{2x^2} = \frac{1}{2} = L_1 \]

Along \( y = x \)
\[ \sin \alpha \frac{x^2}{x^2} = 1 \]

\[ \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} = \lim_{(y,-x) \to (0,0)} \frac{-x^2}{2x^2} = -\frac{1}{2} = L_2 \]

Along \( y = -x \)
\[ \sin \alpha \frac{x^2}{x^2} = -1 \]

Since \( L_1 = \frac{1}{2} \neq L_2 = -\frac{1}{2} \),
\[ \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} \text{ Does Not Exist} \]
Sometimes, for a function \( z = f(x,y) \), we can prove that \( \lim_{(x,y) \to (0,0)} f(x,y) \) exists if we know that \( \lim_{(x,y) \to (0,0)} g(x,y) \) exists for another function \( g(x,y) \).

**The Little Squeeze Theorem**

If \( z = f(x,y) \) and \( z = g(x,y) \) and for all \((x,y) \neq (0,0)\), \( 0 \leq |f(x,y)| \leq g(x,y) \) and \( \lim_{(x,y) \to (0,0)} g(x,y) = 0 \), then \( \lim_{(x,y) \to (0,0)} f(x,y) \) exists and \( \lim_{(x,y) \to (0,0)} f(x,y) = 0 \), too.

The figure to the right should convince you that this is true.

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To prove: \( \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2+y^2}} \) exists and \( \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 \).

**Proof:** Let \( f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \) and \( g(x,y) = |x| \).

\[
|f(x,y)| = \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \frac{|y|}{\sqrt{x^2+y^2}} |x| = |x| \cdot |g(x,y)|
\]

Since \( |y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} \), \( \frac{|y|}{\sqrt{x^2+y^2}} \leq 1 \), \( 0 \leq \frac{|y|}{\sqrt{x^2+y^2}} \leq 1 \).

So, \( 0 \leq |f(x,y)| = \frac{|y|}{\sqrt{x^2+y^2}} |x| \leq 1 \cdot |x| = |x| = g(x,y) \).
(Proof continued)

Also \( \lim_{(x,y) \to (0,0)} |x| = 0 \) since \( x \to 0 \) as \((x,y) \to (0,0)\).

Thus, \( 0 \leq \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq |x| \) \( \int \) Again, because \( \frac{|y|}{\sqrt{x^2+y^2}} \leq 1 \)

and \( \lim_{(x,y) \to (0,0)} |x| = 0 \).

Therefore, by the Little Squeeze Theorem,

\[ \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2+y^2}} \text{ exists and } \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0. \]

Here is another example of a limit calculation:

\[ \lim_{(x,y) \to (1,2)} \frac{x^2+x-2}{xy^2-y^2} = \lim_{(x,y) \to (1,2)} \frac{(x-1)(x+2)}{(x-1)y^2} = \lim_{(x,y) \to (1,2)} \frac{x+2}{y^2} \]

\[ = \frac{3}{4}, \text{ so } \lim_{(x,y) \to (1,2)} \frac{x^2+x-2}{xy^2-y^2} = \frac{3}{4}. \]