LAST TIME – Double RIEMANN Sums

of a function \( z = f(x,y) \)

over a rectangle \( R = [a,b] \times [c,d] \):

RIEMANN Sum = \( \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A_{ij} \).

Then we define the number

\[
\iint_R f(x,y) \, dA = \lim_{m \to \infty} \lim_{n \to \infty} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A_{ij} \right)
\]

How we find this number?

By using \underline{Iterated} Integrals

For Example: Let \( R = [0,3] \times [1,2] \)

\( x' \)'s \( y' \)'s

Let \( f(x,y) = y^2 + 8xy^3 + x^3 \).

How do we determine the number

\( \iint_R (y^2 + 8xy^3 + x^3) \, dA \) ?

Here, it is the Volume of the solid region above rectangle \( R \) and below the surface graph.

Use Iterated Integrals:

Let \( x \) be fixed at a value in \([a,b]\), \( a \leq x \leq b \), and, while \( x \) is held constant, let \( y \) vary in \([c,d]\), \( c \leq y \leq d \).

Then \( G_x(y) = f(x,y) \) defines a function of \( y \) for each \( x \), \( a \leq x \leq b \).
Here, for \( x \) with \( 0 \leq x \leq 3 \), define \( g_x(y) = y^2 + 8xy^3 + x^3 \), that is, \( g_x(y) = f(x,y) \).

Here, \( g_0(y) = y^2 \)

\[ g_1(y) = y^2 + 8y^3 + 1 \]

\[ g_{1.5}(y) = y^2 + 12y^3 + 3.375 \]

\[ g_2(y) = y^2 + 16y^3 + 8 \]

\[ g_3(y) = y^2 + 24y^3 + 27 \]

Fix \( x \) at any number with \( a = 0 \leq x \leq b \), and define the number \( A(x) \) as follows:

\[
A(x) = \int_1^2 (y^2 + 8xy^3 + x^3) \, dy = \int_{C}^{A} g_x(y) \, dy
\]

\[
= \left[ \left( \frac{1}{3}y^3 + 2xy^4 + x^3y \right) \right]_{y=1}^{y=2}
\]

\[
= \left( \frac{8}{3} + 32x + 2x^3 \right) - \left( \frac{1}{3} + 2x + x^3 \right)
\]

\[
A(x) = x^3 + 30x + \frac{7}{3}, \quad 0 \leq x \leq 3.
\]

\( A(x) \) is here the cross-sectional area of the solid \( \mathcal{S} \) cut by the vertical plane through \( (x,0,0) \) and perpendicular to the \( x \)-axis.

The **volume** of the solid \( \mathcal{S} \) is determined by

\[
V = \int_{0}^{3} A(x) \, dx = \int_{0}^{3} \left( x^3 + 30x + \frac{7}{3} \right) \, dx = 162.25
\]

Also,

\[
V = \iint_{R} (y^2 + 8xy^3 + x^3) \, dA = 162.25
\]
By Fubini's Theorem (p. 994)

\[
\iint_R (y^2 + 8xy^3 + x^3) \, dA
\]

Also, by Fubini's Theorem, you can

Switch the order of integration:

\[
\iint_R (y^2 + 8xy^3 + x^3) \, dA = \int_1^2 \left( \int_0^3 (y^2 + 8xy^3 + x^3) \, dx \right) \, dy
\]

Problem: Determine the volume \( V \) of the solid \( S \) below the surface graph of \( z = 16 - x^2 - y^2 \) and above the rectangle \([0, 1] \times [0, 1]\).

Solution: \( V = \iiint_R (16 - x^2 - y^2) \, dA \), so

\[
V = \int_0^1 \int_0^1 (16 - x^2 - y^2) \, dy \, dx
\]

\[
= \int_0^1 \left( \int_0^1 (16 - x^2 - y^2) \, dy \right) \, dx
\]

\[
= \int_0^1 \left( \left[ 16y - x^2y - \frac{1}{3}y^3 \right]_{y=0}^{y=1} \right) \, dx
\]

Fix \( x \) and work from the inside out!
\[ V = \int_0^1 \left( 16 - x^2 - \frac{1}{3} \right) - (0) \, dx \]
\[ = \int_0^1 (16 - x^2 - \frac{1}{3}) \, dx = \int_0^1 \left( \frac{47}{3} - x^2 \right) \, dx \]
\[ = \left[ \frac{47}{3} x - \frac{1}{3} x^3 \right]_0^1 = \left( \frac{47}{3} - \frac{1}{3} \right) - (0-0) \]
\[ V = \frac{46}{3} = 15 \frac{1}{3} \text{ cubic units}. \]

Look ahead and be careful which order you choose to do the integrating in.

If you end up with a difficult integral, consider switching the order of integration.

Ex: \( R = [1, 2] \times [0, \pi] \).

Determine \( \iint_R y \sin(xy) \, dA \).

You have two choices - Which order should you choose?

\[ \int_1^2 \int_0^\pi y \sin(xy) \, dy \, dx \quad \text{OR} \quad \int_0^\pi \int_1^2 y \sin(xy) \, dx \, dy \]

A PRODUCT OF TWO FUNCTIONS OF \( y \) TO INTEGRATE "\( dy \)."

MUST USE INTEGRATION BY PARTS HERE

A PRODUCT OF A CONSTANT TIMES A FUNCTION OF \( x \) TO INTEGRATE "\( dx \)."

You CAN USE U-SUBSTITUTION HERE.

CHOOSE THIS ONE!
Compare these functions:

\[ f_1(x, y) = x^4 \sin y \quad \text{and} \quad f_2(x, y) = x^4 \sin(xy) \]

\[ f_1(x, y) = g(x) h(y) \quad \text{and} \quad f_2(x, y) = x^4 \sin(x y) \]

When \( f(x, y) = g(x) h(y) \),

\[
\int_a^b \int_c^d (g(x) h(y)) \, dy \, dx
\]

\[
= \left( \int_a^b g(x) \, dx \right) \left( \int_c^d h(y) \, dy \right)
\]

because

\[
\int_a^b \left( \int_c^d g(x) h(y) \, dy \right) \, dx
\]

\[
= \int_a^b \left( g(x) \int_c^d h(y) \, dy \right) \, dx
\]

\[
= \int_a^b \left( g(x) \left( h(y) \right)_{y=c}^{y=d} \right) \, dx
\]

\[
= \left( \int_c^d h(y) \, dy \right) \int_a^b g(x) \, dx = \int_a^b \left( \int_c^d f(x) \, dx \right) \, dy
\]

so,

\[
\int_0^1 \int_0^{\pi/2} x^4 \sin y \, dy \, dx = \left( \int_0^1 x^4 \, dx \right) \left( \int_0^{\pi/2} \sin y \, dy \right) = \left( \frac{1}{5} \right) (1) = \frac{1}{5}
\]