TWO PROBLEMS USING
LAGRANGE MULTIPLIERS

TWO PROBLEMS THAT USE LAGRANGE MULTIPLIERS ARE PRESENTED HERE WITH WORKED-OUT SOLUTIONS.

THE FIRST PROBLEM WILL BE PRESENTED IN CLASS, BUT THE COMPLICATED ALGEBRA NEEDED IN WORKING THE PROBLEM MIGHT NOT BE SHOWN DURING THE LECTURE DUE TO A SHORTAGE OF TIME. THAT COMPLICATED ALGEBRA IS PRESENTED HERE.

THE SECOND PROBLEM IS AN EXAMPLE OF A PROBLEM TO FIND A POINT ON A GRAPH THAT IS CLOSEST TO A GIVEN FIXED POINT.
Problem: Let \( z = f(x, y) = xy^2 \).

Determine the max. value of \( f \) and the min. value of \( f \) subject to the constraint \( 9x^2 + 4y^2 = 36 \).

Soln: Maximize/Minimize \( f(x, y) = xy^2 \) subject to \( g(x, y) = 9x^2 + 4y^2 - 36 = 0 \).

\[
\nabla f = \langle y^2, 2xy \rangle, \quad \nabla g = \langle 18x, 8y \rangle
\]

Find all points \((x, y)\) so that a number \( \lambda \) exists such that \( \nabla f(x, y) = \lambda \nabla g(x, y) \) and \( g(x, y) = 0 \).

Evaluate \( f(x, y) \) at all of these points and the largest is the max and the smallest is the min.

\[
\nabla f = \lambda \nabla g \Rightarrow \langle y^2, 2xy \rangle = \lambda \langle 18x, 8y \rangle
\]

We must solve these equations simultaneously:

1. \( y^2 = \lambda (18x) = (18x) \lambda \)
2. \( 2xy = \lambda (8y) \)
3. \( 9x^2 + 4y^2 = 36 \) [Do not forget to include the constraint equation!]

We would like to be able to divide by \( y \) in (2), so we must use cases.

Case 1
Assume \( y = 0 \)

By (3), \( 9x^2 = 36 \)
\( x^2 = 4 \)
\( x = \pm 2, y = 0 \)
Points: \((2, 0), (-2, 0)\)

\( f(2,0) = f(-2,0) = 0 \)

Case 2
Assume \( y \neq 0 \). [Divide (2) by \( y \)]

By (3), \( 2x = 8 \lambda \Rightarrow x = 4 \lambda \Rightarrow x^2 = 16 \lambda^2 \)

Also, with \( x = 4 \lambda \), by (1), \( y^2 = (8 \lambda)(4 \lambda) = 72 \lambda^2 \)

By (3), \( 9(16 \lambda^2) + 4(72 \lambda^2) = 36 \) \[\div 36\]
\( 4 \lambda^2 + 8 \lambda^2 = 1 \Rightarrow 12 \lambda^2 = 1 \)
\( \Rightarrow \lambda^2 = \frac{1}{12} = \frac{3}{36} \Rightarrow \lambda = \pm \frac{\sqrt{3}}{6} \)
Case 2: \( y \neq 0 \) (cont.)

Recall \( \lambda = \pm \frac{\sqrt{3}}{6} \) and \( x = 4\lambda \) and \( y^2 = 72\lambda^2 \)

So, \( x = \pm \frac{4\sqrt{3}}{6} \Rightarrow x = \pm \frac{2\sqrt{3}}{3} \)

\[ \Rightarrow (x,y) = (\pm \frac{2\sqrt{3}}{3}, \pm \sqrt{6}) \]

and \( y^2 = 72 \left( \frac{3}{36} \right) = 6 \Rightarrow y = \pm \sqrt{6} \)

with \( f(x,y) = xy^2 \)

\[ f\left( \frac{2\sqrt{3}}{3}, \sqrt{6} \right) = \frac{2\sqrt{3}}{3} \times 6 = 4\sqrt{3} \approx 6.93 \]

\[ f\left( \frac{2\sqrt{3}}{3}, -\sqrt{6} \right) = \frac{2\sqrt{3}}{3} \times 6 = 4\sqrt{3} \]

\[ f\left( -\frac{2\sqrt{3}}{3}, \sqrt{6} \right) = -\frac{2\sqrt{3}}{3} \times 6 = -4\sqrt{3} \approx -6.93 \]

\[ f\left( -\frac{2\sqrt{3}}{3}, -\sqrt{6} \right) = -\frac{2\sqrt{3}}{3} \times 6 = -4\sqrt{3} \]

Also \( f(-2,0) = 0 \) from Case 1.

\[ f(2,0) = 0 \]

The maximum value of \( f \) subject to the constraint \( g(xy) = 0 \) is \( 4\sqrt{3} = f\left( \frac{2\sqrt{3}}{3}, \sqrt{6} \right) \).

The minimum value of \( f \) subject to the constraint \( g(xy) = 0 \) is \( -4\sqrt{3} = f\left( -\frac{2\sqrt{3}}{3}, \sqrt{6} \right) \).
Problem: Find the point on the hyperbola branch $x^2 - y^2 = 1, \ x > 0$, which is closest to the point $(0, 2)$.

Solution:

Minimize $d = \sqrt{x^2 + (y-2)^2}$

Minimize

$$d^2 = f(xy) = x^2 + (y-2)^2$$

Constraint $g(xy) = x^2 - y^2 - 1 = 0, \ x > 0$

Since there is no maximum value of $f(xy) = d^2$ and only one point $(x_0, y_0)$ with minimum distance $d$, we will probably find only one point with a number $\lambda$ so that $\nabla f = \lambda \nabla g$.

$$f(xy) = x^2 + (y-2)^2 \quad g(xy) = x^2 - y^2 - 1$$

$$\nabla f = \langle 2x, 2(y-2) \rangle \quad \nabla g = \langle 2x, -2y \rangle$$

Solve for $\lambda$

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2(y-2) \rangle = \lambda \langle 2x, -2y \rangle$$

1. $2x = \lambda 2x$
2. $2(y-2) = \lambda (-2y)$

[Don't forget to include the constraint equation (3)]

By (3), $x \neq 0$; So, by dividing by $2x$ in (1), $\lambda = 1$

By (2) with $\lambda = 1$, $2y-4 = -2y \Rightarrow 4y = 4$,

$\Rightarrow y = 1$. \Rightarrow By (3), $x^2 - 1 = 1 \Rightarrow x^2 = 2 \Rightarrow x = \pm \sqrt{2}$

So, $y = 1$. \Rightarrow By (3), $x^2 = 1 \Rightarrow x = \pm 1$

Since $x > 0$, $x = \sqrt{2}$. The only point with minimum $d^2$ is $(x_0, y_0) = (\sqrt{2}, 1)$.

The point on the branch of the hyperbola $x^2 - y^2 = 1$ which is closest to the point $(0, 2)$ is $(x_0, y_0) = (\sqrt{2}, 1)$.